

# Combinatorics Through Guided Discovery<sup>1</sup>

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# Preface

This book is an introduction to combinatorial mathematics, also known as combinatorics. The book focuses especially but not exclusively on the part of combinatorics that mathematicians refer to as “counting.” The book consists almost entirely of problems. Some of the problems are designed to lead you to think about a concept, others are designed to help you figure out a concept and state a theorem about it, while still others ask you to prove the theorem. Other problems give you a chance to use a theorem you have proved. From time to time there is a discussion that pulls together some of the things you have learned or introduces a new idea for you to work with. Many of the problems are designed to build up your intuition for how combinatorial mathematics works. There are problems that some people will solve quickly, and there are problems that will take days of thought for everyone. Probably the best way to use this book is to work on a problem until you feel you are not making progress and then go on to the next one. Think about the problem you couldn’t get as you do other things. The next chance you get, discuss the problem you are stymied on with other members of the class. Often you will all feel you’ve hit dead ends, but when you begin comparing notes and listening *carefully* to each other, you will see more than one approach to the problem and be able to make some progress. In fact, after comparing notes you may realize that there is more than one way to interpret the problem. In this case your first step should be to think together about what the problem is actually asking you to do. You may have learned in school that for every problem you are given, there is a method that has already been taught to you, and you are supposed to figure out which method applies and apply it. That is not the case here. Based on some simplified examples, you will discover the method for yourself. Later on, you may recognize a pattern that suggests you should try to use this method again.

The point of learning from this book is that you are learning how to discover ideas and methods for yourself, not that you are learning to apply

Table 1: The meaning of the symbols to the left of problem numbers.

•	essential
○	motivational material
+	summary
➔	especially interesting
*	difficult
·	essential for this or the next section

methods that someone else has told you about. The problems in this book are designed to lead you to discover for yourself and prove for yourself the main ideas of combinatorial mathematics. There is considerable evidence that this leads to deeper learning and more understanding.

You will see that some of the problems are marked with bullets. Those are the problems that I feel are essential to having an understanding of what comes later, whether or not it is marked by a bullet. The problems with bullets are the problems in which the main ideas of the book are developed. Your instructor may leave out some of these problems because he or she plans not to cover future problems that rely on them. Many problems, in fact entire sections, are not marked in this way, because they use an important idea rather than developing one. Some other special symbols are described in what follows; a summary appears in Table 1.

Some problems are marked with open circles. This indicates that they are designed to provide motivation for, or an introduction to, the important concepts, motivation with which some students may already be familiar. You will also see that some problems are marked with arrows. These point to problems that I think are particularly interesting. Some of them are also difficult, but not all are. A few problems that summarize ideas that have come before but aren't really essential are marked with a plus, and problems that are essential if you want to cover the section they are in or, perhaps, the next section, are marked with a dot (a small bullet). If a problem is relevant to a much later section in an essential way, I've marked it with a dot and a parenthetical note that explains where it will be essential. Finally, problems that seem unusually hard to me are marked with an asterisk. Some I've marked as hard only because I think they are difficult in light of what has come before, not because they are intrinsically difficult. In particular, some of the problems marked as hard will not seem so hard if you come back to them after you have finished more of the problems.



If you are taking a course, your instructor will choose problems for you to work on based on the prerequisites for and goals of the course. If you are reading the book on your own, I recommend that you try all the problems in a section you want to cover. Try to do the problems with bullets, but by all means don't restrict yourself to them. Often a bulleted problem makes more sense if you have done some of the easier motivational problems that come before it. If, after you've tried it, you want to skip over a problem without a bullet or circle, you should not miss out on much by not doing that problem. Also, if you don't find the problems in a section with no bullets interesting, you can skip them, understanding that you may be skipping an entire branch of combinatorial mathematics! And no matter what, read the textual material that comes before, between, and immediately after problems you are working on!

One of the downsides of how we learn math in high school is that many of us come to believe that if we can't solve a problem in ten or twenty minutes, then we can't solve it at all. There will be problems in this book that take hours of hard thought. Many of these problems were first conceived and solved by professional mathematicians, and *they* spent days or weeks on them. How can you be expected to solve them at all then? You have a context in which to work, and even though some of the problems are so open ended that you go into them without any idea of the answer, the context and the leading examples that precede them give you a structure to work with. That doesn't mean you'll get them right away, but you will find a real sense of satisfaction when you see what you can figure out with concentrated thought. Besides, you can get hints!

Some of the questions will appear to be trick questions, especially when you get the answer. They are not intended as trick questions at all. Instead they are designed so that they don't tell you the answer in advance. For example the answer to a question that begins "How many..." might be "none." Or there might be just one example (or even no examples) for a problem that asks you to find all examples of something. So when you read a question, unless it directly tells you what the answer is and asks you to show it is true, don't expect the wording of the problem to suggest the answer. The book isn't designed this way to be cruel. Rather, there is evidence that the more open-ended a question is, the more deeply you learn from working on it. If you do go on to do mathematics later in life, the problems that come to you from the real world or from exploring a mathematical topic are going to be open-ended problems because nobody will have done them before. Thus working on open-ended problems now should help to prepare you to do mathematics and apply mathematics in other areas later on.

You should try to write up answers to all the problems that you work on. If you claim something is true, you should explain why it is true; that is you should prove it. In some cases an idea is introduced before you have the tools to prove it, or the proof of something will add nothing to your understanding. In such problems there is a remark telling you not to bother with a proof. When you write up a problem, remember that the instructor has to be able to “get” your ideas and understand exactly what you are saying. Your instructor is going to choose some of your solutions to read carefully and give you detailed feedback on. When you get this feedback, you should think it over carefully and then write the solution again! You may be asked not to have someone else read your solutions to some of these problems until your instructor has. This is so that the instructor can offer help which is aimed at your needs. On other problems it is a good idea to seek feedback from other students. One of the best ways of learning to write clearly is to have someone point out to you where it is hard to figure out what you mean. The crucial thing is to make it clear to your reader that you really want to know where you may have left something out, made an unclear statement, or failed to support a statement with a proof. It is often very helpful to choose people who have not yet become an expert with the problems, as long as they realize it will help you most for them to tell you about places in your solutions they do not understand, even if they think it is their problem and not yours!

As you work on a problem, think about why you are doing what you are doing. Is it helping you? If your current approach doesn't feel right, try to see why. Is this a problem you can decompose into simpler problems? Can you see a way to make up a simple example, even a silly one, of what the problem is asking you to do? If a problem is asking you to do something for every value of an integer  $n$ , then what happens with simple values of  $n$  like 0, 1, and 2? Don't worry about making mistakes; it is often finding mistakes that leads mathematicians to their best insights. Above all, don't worry if you can't do a problem. Some problems are given as soon as there is one technique you've learned that might help do that problem. Later on there may be other techniques that you can bring back to that problem to try again. The notes have been designed this way on purpose. If you happen to get a hard problem with the bare minimum of tools, you will have accomplished much. As you go along, you will see your ideas appearing again later in other problems. On the other hand, if you don't get the problem the first time through, it will be nagging at you as you work on other things, and when you see the idea for an old problem in new work, you will know you are learning.

There are quite a few concepts that are developed in this book. Since most of the intellectual content is in the problems, it is natural that definitions of concepts will often be within problems. When you come across an unfamiliar term in a problem, it is likely it was defined earlier. Look it up in the index, and with luck (hopefully no luck will really be needed!) you will be able to find the definition.

Above all, this book is dedicated to the principle that doing mathematics is fun. As long as you know that some of the problems are going to require more than one attempt before you hit on the main idea, you can relax and enjoy your successes, knowing that as you work more and more problems and share more and more ideas, problems that seemed intractable at first become a source of satisfaction later on.

The development of this book is supported by the National Science Foundation. An essential part of this support is an advisory board of faculty members from a wide variety of institutions who have tried to help me understand what would make the book helpful in their institutions. They are Karen Collins, Wesleyan University, Marc Lipman, Indiana University/Purdue University, Fort Wayne, Elizabeth MacMahon, Lafayette College, Fred McMorris, Illinois Institute of Technology, Mark Miller, Marietta College, Rosa Orellana, Dartmouth College, Vic Reiner, University of Minnesota, and Lou Shapiro, Howard University. The overall design and most of the problems in the appendix on exponential generating functions are due to Professors Reiner and Shapiro. Any errors or confusing writing in that appendix are due to me! I believe the board has managed both to make the book more accessible and more interesting.



# Chapter 1

## What is Combinatorics?

Combinatorial mathematics arises from studying how we can *combine* objects into arrangements. For example, we might be combining sports teams into a tournament, samples of tires into plans to mount them on cars for testing, students into classes to compare approaches to teaching a subject, or members of a tennis club into pairs to play tennis. There are many questions one can ask about such arrangements of objects. Here we will focus on questions about *how many ways* we may combine the objects into arrangements of the desired type. These are called *counting problems*. Sometimes, though, combinatorial mathematicians ask if an arrangement is possible (if we have ten baseball teams, and each team has to play each other team once, can we schedule all the games if we only have the fields available at enough times for forty games?). Sometimes they ask if all the arrangements we might be able to make have a certain desirable property (Do all ways of testing 5 brands of tires on 5 different cars [with certain additional properties] compare each brand with each other brand on at least one common car?). Counting problems (and problems of the other sorts described) come up throughout physics, biology, computer science, statistics, and many other subjects. However, to demonstrate all these relationships, we would have to take detours into all these subjects. While we will give some important applications, we will usually phrase our discussions around everyday experience and mathematical experience so that the student does not have to learn a new context before learning mathematics in context!

## 1.1 About These Notes

These notes are based on the philosophy that you learn the most about a subject when you are figuring it out directly for yourself, and learn the least when you are trying to figure out what someone else is saying about it. On the other hand, there is a subject called combinatorial mathematics, and that is what we are going to be studying, so we will have to tell you some basic facts. What we are going to try to do is to give you a chance to discover many of the interesting examples that usually appear as textbook examples and discover the principles that appear as textbook theorems. Your main activity will be solving problems designed to lead you to discover the basic principles of combinatorial mathematics. Some of the problems lead you through a new idea, some give you a chance to describe what you have learned in a sequence of problems, and some are quite challenging. When you find a problem challenging, don't give up on it, but don't let it stop you from going on with other problems. Frequently you will find an idea in a later problem that you can take back to the one you skipped over or only partly finished in order to finish it off. With that in mind, let's get started. In the problems that follow, you will see some problems marked on the left with various symbols. The preface gives a full explanation of these symbols and discusses in greater detail why the book is organized as it is! Table 1.1, which is repeated from the preface, summarizes the meaning of the symbols.

Table 1.1: The meaning of the symbols to the left of problem numbers.

•	essential
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+	summary
➔	especially interesting
*	difficult
·	essential for this or the next section

## 1.2 Basic Counting Principles

- 1. Five schools are going to send their baseball teams to a tournament, in which each team must play each other team exactly once. How many games are required?

- 2. Now some number  $n$  of schools are going to send their baseball teams to a tournament, and each team must play each other team exactly once. Let us think of the teams as numbered 1 through  $n$ .
  - (a) How many games does team 1 have to play in?
  - (b) How many games, other than the one with team 1, does team two have to play in?
  - (c) How many games, other than those with the first  $i - 1$  teams, does team  $i$  have to play in?
  - (d) In terms of your answers to the previous parts of this problem, what is the total number of games that must be played?
- 3. One of the schools sending its team to the tournament has to send its players from some distance, and so it is making sandwiches for team members to eat along the way. There are three choices for the kind of bread and five choices for the kind of filling. How many different kinds of sandwiches are available?
- +4. An *ordered pair*  $(a, b)$  consists of two things we call  $a$  and  $b$ . We say  $a$  is the first member of the pair and  $b$  is the second member of the pair. If  $M$  is an  $m$ -element set and  $N$  is an  $n$ -element set, how many ordered pairs are there whose first member is in  $M$  and whose second member is in  $N$ ? Does this problem have anything to do with any of the previous problems?
- 5. Since a sandwich by itself is pretty boring, students from the school in Problem 3 are offered a choice of a drink (from among five different kinds), a sandwich, and a fruit (from among four different kinds). In how many ways may a student make a choice of the three items now?
- 6. The coach of the team in Problem 3 knows of an ice cream parlor along the way where she plans to stop to buy each team member a triple decker cone. There are 12 different flavors of ice cream, and triple decker cones are made in homemade waffle cones. Having chocolate ice cream as the bottom scoop is different from having chocolate ice cream as the top scoop. How many possible ice cream cones are going to be available to the team members? How many cones with three different kinds of ice cream will be available?
- 7. The idea of a function is ubiquitous in mathematics. A function  $f$  from a set  $S$  to a set  $T$  is a relationship between the two sets that associates

exactly one member  $f(x)$  of  $T$  with each element  $x$  in  $S$ . We will come back to the ideas of functions and relationships in more detail and from different points of view from time to time. However, the quick review above should probably let you answer these questions. If you have difficulty with them, it would be a good idea to go now to Appendix A and work through Section A.1.1 which covers this definition in more detail. You might also want to study Section A.1.3 to learn to visualize the properties of functions. We will take up the topic of this section later in this chapter as well, but in less detail than is in the appendix.

- (a) Using  $f, g, \dots$ , to stand for the various functions, write down all the different functions you can from the set  $\{1, 2\}$  to the set  $\{a, b\}$ . For example, you might start with the function  $f$  given by  $f(1) = a, f(2) = b$ . How many functions are there from the set  $\{1, 2\}$  to the set  $\{a, b\}$ ?
  - (b) How many functions are there from the three element set  $\{1, 2, 3\}$  to the two element set  $\{a, b\}$ ?
  - (c) How many functions are there from the two element set  $\{a, b\}$  to the three element set  $\{1, 2, 3\}$ ?
  - (d) How many functions are there from a three element set to a 12 element set?
  - (e) A function  $f$  is called **one-to-one** or an *injection* if whenever  $x$  is different from  $y$ ,  $f(x)$  is different from  $f(y)$ . How many one-to-one functions are there from a three element set to a 12 element set?
  - (f) Explain the relationship between this problem and Problem 6.
- 8. A group of hungry team members in Problem 6 notices it would be cheaper to buy three pints of ice cream for them to split than to buy a triple decker cone for each of them, and that way they would get more ice cream. They ask their coach if they can buy three pints of ice cream.
    - (a) In how many ways can they choose three pints of different flavors out of the 12 flavors?
    - (b) In how many ways may they choose three pints if the flavors don't have to be different?
  - 9. Two sets are said to be *disjoint* if they have no elements in common. For example,  $\{1, 3, 12\}$  and  $\{6, 4, 8, 2\}$  are disjoint, but  $\{1, 3, 12\}$  and



$\{3, 5, 7\}$  are not. Three or more sets are said to be *mutually disjoint* if no two of them have any elements in common. What can you say about the size of the union of a finite number of finite (mutually) disjoint sets? Does this have anything to do with any of the previous problems?

- 10. Disjoint subsets are defined in Problem 9. What can you say about the size of the union of  $m$  (mutually) disjoint sets, each of size  $n$ ? Does this have anything to do with any of the previous problems?

### 1.2.1 The sum and product principles

These problems contain among them the kernels of many of the fundamental ideas of combinatorics. For example, with luck, you just stated the sum principle (illustrated in Figure 1.1), and product principle (illustrated in Figure 1.2) in Problems 9 and 10. These are two of the most basic principles of combinatorics. These two counting principles are the basis on which we will develop many other counting principles.

Figure 1.1: The union of these two disjoint sets has size 17.

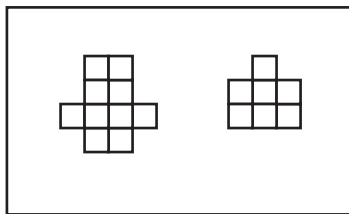
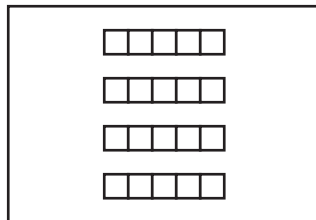


Figure 1.2: The union of four disjoint sets of size five.



You may have noticed some standard mathematical words and phrases such as *set*, *ordered pair*, *function* and so on creeping into the problems. One of our goals in these notes is to show how most counting problems can be recognized as counting all or some of the elements of a set of standard mathematical objects. For example, Problem 4 is meant to suggest that the question we asked in Problem 3 was really a problem of counting all the ordered pairs consisting of a bread choice and a filling choice. We use  $A \times B$  to stand for the set of all ordered pairs whose first element is in  $A$  and whose second element is in  $B$  and we call  $A \times B$  the *Cartesian product* of  $A$  and  $B$ . Thus you can think of Problem 4 as asking you for the size of the Cartesian product of  $M$  and  $N$ , that is, asking you to count the number of elements of this Cartesian product.

When a set  $S$  is a union of disjoint sets  $B_1, B_2, \dots, B_m$  we say that the sets  $B_1, B_2, \dots, B_m$  are a **partition** of the set  $S$ . Thus a partition of  $S$  is a (special kind of) set of sets. So that we don't find ourselves getting confused between the set  $S$  and the sets  $B_i$  into which we have divided it, we often call the sets  $B_1, B_2, \dots, B_m$  the *blocks* of the partition. In this language, the **sum principle** says that

if we have a partition of a set  $S$ , then the size of  $S$  is the sum of the sizes of the blocks of the partition.

The **product principle** says that

if we have a partition of a set  $S$  into  $m$  blocks, each of size  $n$ , then  $S$  has size  $mn$ .

There is another version of the product principle that applies directly in problems like Problem 5 and Problem 6, where we were not just taking a union of  $m$  disjoint sets of size  $n$ , but rather  $m$  disjoint sets of size  $n$ , each of which was a union of  $m'$  disjoint sets of size  $n'$ . This is an inconvenient way to have to think about a counting problem, so we may rephrase the product principle in terms of a sequence of decisions:

- 11. If we make a sequence of  $m$  choices for which
  - there are  $k_1$  possible first choices, and
  - for each way of making the first  $i - 1$  choices, there are  $k_i$  ways to make the  $i$ th choice,

then in how many ways may we make our sequence of choices? (You need not prove your answer correct at this time.)

The counting principle you gave in Problem 11 is called the *general product principle*. We will outline a proof of the general product principle from the original product principle in Problem 79. Until then, let us simply accept it as another counting principle. For now, notice how much easier it makes it to explain why we multiplied the things we did in Problem 5 and Problem 6.

- 12. A tennis club has  $2n$  members. We want to pair up the members by twos for singles matches.
- (a) In how many ways may we pair up all the members of the club? (Hint: consider the cases of 2, 4, and 6 members.)
  - (b) Suppose that in addition to specifying who plays whom, for each pairing we say who serves first. Now in how many ways may we specify our pairs?
- + 13. Let us now return to Problem 7 and justify—or perhaps finish—our answer to the question about the number of functions from a three-element set to a 12-element set.
- (a) How can you justify your answer in Problem 7 to the question “How many functions are there from a three element set (say [3]) to a twelve element set (say [12])? ”
  - (b) Based on the examples you’ve seen so far, make a conjecture about how many functions there are from  $[m]$  to  $[n]$  and prove it.
  - (c) A common notation for the set of all functions from a set  $M$  to a set  $N$  is  $N^M$ . Why is this a good notation?
- + 14. Now suppose we are thinking about a set  $S$  of functions  $f$  from  $[m]$  to some set  $X$ . (For example, in Problem 6 we were thinking of the set of functions from the three possible places for scoops in an ice-cream cone to 12 flavors of ice cream.) Suppose there are  $k_1$  choices for  $f(1)$ . (In Problem 6,  $k_1$  was 12, because there were 12 ways to choose the first scoop.) Suppose that for each choice of  $f(1)$  there are  $k_2$  choices for  $f(2)$ . (For example, in Problem 6  $k_2$  was 12 if the second flavor could be the same as the first, but  $k_2$  was 11 if the flavors had to be different.) In general, suppose that for each choice of  $f(1)$ ,  $f(2)$ ,  $\dots$ ,  $f(i-1)$ , there are  $k_i$  choices for  $f(i)$ . (For example, in Problem 6, if the flavors have to be different, then for each choice of  $f(1)$  and  $f(2)$ , there are 10 choices for  $f(3)$ .)

What we have assumed so far about the functions in  $S$  may be summarized as

- There are  $k_1$  choices for  $f(1)$ .
- For each choice of  $f(1), f(2), \dots, f(i-1)$ , there are  $k_i$  choices for  $f(i)$ .

How many functions are in the set  $S$ ? Is there any practical difference between the result of this problem and the general product principle?

The point of Problem 14 is that the general product principle can be stated informally, as we did originally, or as a statement about counting sets of standard concrete mathematical objects, namely functions.

- 15. A roller coaster car has  $n$  rows of seats, each of which has room for two people. If  $n$  men and  $n$  women get into the car with a man and a woman in each row, in how many ways may they choose their seats?
- + 16. How does the general product principle apply to Problem 6?
17. In how many ways can we pass out  $k$  distinct pieces of fruit to  $n$  children (with no restriction on how many pieces of fruit a child may get)?
- 18. How many subsets does a set  $S$  with  $n$  elements have?
- 19. Assuming  $k \leq n$ , in how many ways can we pass out  $k$  distinct pieces of fruit to  $n$  children if each child may get at most one? What is the number if  $k > n$ ? Assume for both questions that we pass out all the fruit.
- 20. Another name for a list, in a specific order, of  $k$  distinct things chosen from a set  $S$  is a **k-element permutation of S**. We can also think of a  $k$ -element permutation of  $S$  as a one-to-one function (or, in other words, injection) from  $[k] = \{1, 2, \dots, k\}$  to  $S$ . How many  $k$ -element permutations does an  $n$ -element set have? (For this problem it is natural to assume  $k \leq n$ . However, the question makes sense even if  $k > n$ .) What is the number of  $k$ -element permutations of an  $n$ -element set if  $k > n$ ?

There are a variety of different notations for the number of  $k$ -element permutations of an  $n$ -element set. The one we shall use was introduced by

Don Knuth; namely  $n^{\underline{k}}$ , read “ $n$  to the  $k$  falling” or “ $n$  to the  $k$  down.” In Problem 20 you may have shown that

$$n^{\underline{k}} = n(n-1)\cdots(n-k+1) = \prod_{i=1}^k (n-i+1). \quad (1.1)$$

It is standard to call  $n^{\underline{k}}$  the  **$k$ -th falling factorial power of  $n$** , which explains why we use exponential notation. We call it a *factorial* power since  $n^{\underline{n}} = n(n-1)\cdots 1$ , which we call  *$n$ -factorial* and denote by  $n!$ . If you are unfamiliar with the Pi notation, or *product notation* we introduced for products in Equation 1.1, it works just like the Sigma notation works for summations.

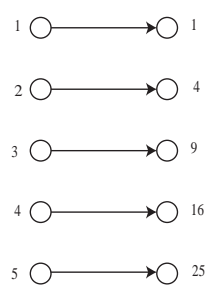
- 21. Express  $n^{\underline{k}}$  as a quotient of factorials.

### 1.2.2 Functions and directed graphs

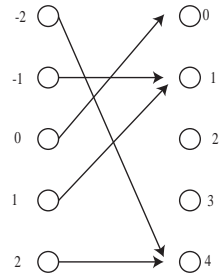
As another example of how standard mathematical language relates to counting problems, Problem 7 explicitly asked you to relate the idea of counting functions to the question of Problem 6. You have probably learned in algebra or calculus how to draw graphs in the cartesian plane of functions from a set of numbers to a set of numbers. You may recall how we can determine whether a graph is a graph of a function by examining whether each vertical straight line crosses the graph at most one time. You might also recall how we can determine whether such a function is one-to-one by examining whether each horizontal straight line crosses the graph at most one time. The functions we deal with will often involve objects which are not numbers, and will often be functions from one finite set to another. Thus graphs in the cartesian plane will often not be available to us for visualizing functions.

However, there is another kind of graph called a *directed graph* or *digraph* that is especially useful when dealing with functions between finite sets. We take up this topic in more detail in Appendix A, particularly Section A.1.2 and Section A.1.3. In Figure 1.3 we show several examples of digraphs of functions. If we have a function  $f$  from a set  $S$  to a set  $T$ , we draw a line of dots or circles, called *vertices* to represent the elements of  $S$  and another (usually parallel) line of vertices to represent the elements of  $T$ . We then draw an arrow from the vertex for  $x$  to the vertex for  $y$  if  $f(x) = y$ . Sometimes, as in part (e) of the figure, if we have a function from a set  $S$  to itself, we draw only one set of vertices representing the elements of  $S$ , in which case we can have arrows both entering and leaving a given vertex. As

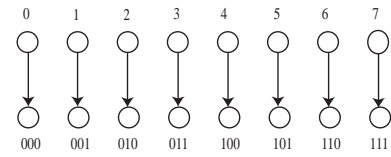
Figure 1.3: What is a digraph of a function?



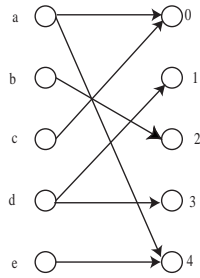
(a) The function given by  $f(x) = x^2$  on the domain  $\{1, 2, 3, 4, 5\}$ .



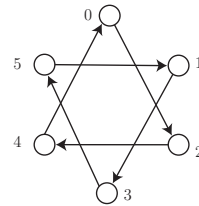
(c) The function from the set  $\{-2, -1, 0, 1, 2\}$  to the set  $\{0, 1, 2, 3, 4\}$  given by  $f(x) = x^2$ .



(b) The function from the set  $\{0, 1, 2, 3, 4, 5, 6, 7\}$  to the set of triples of zeros and ones given by  $f(x)$  = the binary representation of  $x$ .



(d) Not the digraph of a function.



(e) The function from  $\{0, 1, 2, 3, 4, 5\}$  to  $\{0, 1, 2, 3, 4, 5\}$  given by  $f(x) = x + 2 \pmod 6$

you see, the digraph can be more enlightening in this case if we experiment with the function to find a nice placement of the vertices rather than putting them in a row.

Notice that there is a simple test for whether a digraph whose vertices represent the elements of the sets  $S$  and  $T$  is the digraph of a function from  $S$  to  $T$ . There must be one and only one arrow leaving each vertex of the digraph representing an element of  $S$ . The fact that there is one arrow means that  $f(x)$  is defined for each  $x$  in  $S$ . The fact that there is only one arrow means that each  $x$  in  $S$  is related to exactly one element of  $T$ . (Note that these remarks hold as well if we have a function from  $S$  to  $S$  and draw only one set of vertices representing the elements of  $S$ .) For further discussion of functions and digraphs see Sections A.1.1 and A.1.2 of Appendix A.

- 22. Draw the digraph of the function from the set {Alice, Bob, Dawn, Bill} to the set {A, B, C, D, E} given by

$$f(X) = \text{the first letter of the name } X.$$

- 23. A function  $f : S \rightarrow T$  is called an *onto function* or *surjection* if each element of  $T$  is  $f(x)$  for some  $x \in S$ . Choose a set  $S$  and a set  $T$  so that you can draw the digraph of a function from  $S$  to  $T$  that is one-to-one but not onto, and draw the digraph of such a function.
- 24. Choose a set  $S$  and a set  $T$  so that you can draw the digraph of a function from  $S$  to  $T$  that is onto but not one-to-one, and draw the digraph of such a function.
- 25. Digraphs of functions help us visualize the ideas of one-to-one functions and onto functions.
  - (a) What does the digraph of a one-to-one function (injection) from a finite set  $X$  to a finite set  $Y$  look like? (Look for a test somewhat similar to the one we described for when a digraph is the digraph of a function.)
  - (b) What does the digraph of an onto function look like?
  - (c) What does the digraph of a one-to-one and onto function from a finite set  $S$  to a set  $T$  look like?
- 26. The word *permutation* is actually used in two different ways in mathematics. A **permutation of a set**  $S$  is a one-to-one function from  $S$  onto  $S$ . How many permutations does an  $n$ -element set have?

Notice that there is a great deal of consistency between the use of the word permutation in Problem 26 and the use in the Problem 20. If we have some way  $a_1, a_2, \dots, a_n$  of listing our set  $S$ , then any other list  $b_1, b_2, \dots, b_n$  gives us the permutation of  $S$  whose rule is  $f(a_i) = b_i$ , and any permutation of  $S$ , say the one given by  $g(a_i) = c_i$  gives us a list  $c_1, c_2, \dots, c_n$  of  $S$ . Thus there is really very little difference between the idea of a permutation of  $S$  and an  $n$ -element permutation of  $S$  when  $n$  is the size of  $S$ .

### 1.2.3 The bijection principle

Another name for a one-to-one and onto function is **bijection**. The digraphs marked (a), (b), and (e) in Figure 1.3 are digraphs of bijections. The description in Problem 25c of the digraph of a bijection from  $X$  to  $Y$  illustrates one of the fundamental principles of combinatorial mathematics, the **bijection principle**:

Two sets have the same size if and only if there is a bijection between them.

It is surprising how this innocent sounding principle guides us into finding insight into some otherwise very complicated proofs.

### 1.2.4 Counting subsets of a set

27. The *binary* representation of a number  $m$  is a list, or string,  $a_1 a_2 \dots a_k$  of zeros and ones such that  $m = a_1 2^{k-1} + a_2 2^{k-2} + \dots + a_k 2^0$ . Describe a bijection between the binary representations of the integers between 0 and  $2^n - 1$  and the subsets of an  $n$ -element set. What does this tell you about the number of subsets of the  $n$ -element set  $[n]$ ?

Notice that the first question in Problem 8 asked you for the number of ways to choose a three element subset from a 12 element subset. You may have seen a notation like  $\binom{n}{k}$ ,  $C(n, k)$ , or  ${}_n C_k$  which stands for the number of ways to choose a  $k$ -element subset from an  $n$ -element set. The number  $\binom{n}{k}$  is read as “ $n$  choose  $k$ ” and is called a **binomial coefficient** for reasons we will see later on. Another frequently used way to read the binomial coefficient notation is “the number of combinations of  $n$  things taken  $k$  at a time.” We won’t use this way of reading the notation. You are going to be asked to construct two bijections that relate to these numbers and figure out what famous formula they prove. We are going to think about subsets of the



$n$ -element set  $[n] = \{1, 2, 3, \dots, n\}$ . As an example, the set of two-element subsets of  $[4]$  is

$$\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

This example tells us that  $\binom{4}{2} = 6$ .

- 28. Let  $C$  be the set of  $k$ -element subsets of  $[n]$  that contain the number  $n$ , and let  $D$  be the set of  $k$ -element subsets of  $[n]$  that don't contain  $n$ .
  - (a) Let  $C'$  be the set of  $(k - 1)$ -element subsets of  $[n - 1]$ . Describe a bijection from  $C$  to  $C'$ . (A verbal description is fine.)
  - (b) Let  $D'$  be the set of  $k$ -element subsets of  $[n - 1] = \{1, 2, \dots, n - 1\}$ . Describe a bijection from  $D$  to  $D'$ . (A verbal description is fine.)
  - (c) Based on the two previous parts, express the sizes of  $C$  and  $D$  in terms of binomial coefficients involving  $n - 1$  instead of  $n$ .
  - (d) Apply the sum principle to  $C$  and  $D$  and obtain a formula that expresses  $\binom{n}{k}$  in terms of two binomial coefficients involving  $n - 1$ . You have just derived the Pascal Equation that is the basis for the famous Pascal's Triangle.

### 1.2.5 Pascal's Triangle

The Pascal Equation that you derived in Problem 28 gives us the triangle in Figure 1.4. This figure has the number of  $k$ -element subsets of an  $n$ -element set as the  $k$ th number over in the  $n$ th row (we call the top row the zeroth row and the beginning entry of a row the zeroth number over). You'll see that your formula doesn't say anything about  $\binom{n}{k}$  if  $k = 0$  or  $k = n$ , but otherwise it says that each entry is the sum of the two that are above it and just to the left or right.

- 29. Just for practice, what is the next row of Pascal's triangle?
- 30. Without writing out the rows completely, write out enough of Pascal's triangle to get a numerical answer for the first question in Problem 8.

It is less common to see Pascal's triangle as a right triangle, but it actually makes your formula easier to interpret. In Pascal's Right Triangle, the element in row  $n$  and column  $k$  (with the convention that the first row is row zero and the first column is column zero) is  $\binom{n}{k}$ . In this case your

Figure 1.4: Pascal's Triangle

				1					
				1		1			
			1		2		1		
		1		3		3		1	
	1		4		6		4		1
	1	5		10		10		5	1
	1	6	15		20		15	6	1
1	7	21	35		35		21	7	1

formula says each entry in a row is the sum of the one above and the one above and to the left, except for the leftmost and right most entries of a row, for which that doesn't make sense. Since the leftmost entry is  $\binom{n}{0}$  and the rightmost entry is  $\binom{n}{n}$ , these entries are both one (to see why, ask yourself how many 0-element subsets and how many  $n$ -element subsets an  $n$ -element set has), and your formula then tells how to fill in the rest of the table.

Figure 1.5: Pascal's Right Triangle

	$k = 0$	1	2	3	4	5	6	7
$n = 0$	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

Seeing this right triangle leads us to ask whether there is some natural way to extend the right triangle to a rectangle. If we did have a rectangular table of binomial coefficients, counting the first row as row zero (i.e.,  $n = 0$ ) and the first column as column zero (i.e.,  $k = 0$ ), the entries we don't yet have are values of  $\binom{n}{k}$  for  $k > n$ . But how many  $k$ -element subsets does an  $n$ -element set have if  $k > n$ ? The answer, of course, is zero, so all the other entries we would fill in would be zero, giving us the rectangular array in Figure 1.6. It is straightforward to check that Pascal's Equation now works

for all the entries in the rectangle that have an entry above them and an entry above and to the left.

Figure 1.6: Pascal's Rectangle

	$k = 0$	1	2	3	4	5	6	7
$n = 0$	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	1	2	1	0	0	0	0	0
3	1	3	3	1	0	0	0	0
4	1	4	6	4	1	0	0	0
5	1	5	10	10	5	1	0	0
6	1	6	15	20	15	6	1	0
7	1	7	21	35	35	21	7	1

- 31. Because our definition told us that  $\binom{n}{k}$  is 0 when  $k > n$ , we got a rectangular table of numbers that satisfies the Pascal Equation.
- Is there any other way to define  $\binom{n}{k}$  when  $k > n$  in order to get a rectangular table that agrees with Pascal's Right Triangle for  $k \leq n$  and satisfies the Pascal Equation?
  - Suppose we want to extend Pascal's Rectangle to the left and define  $\binom{n}{-k}$  for  $n \geq 0$  and  $k > 0$  so that  $-k < 0$ . What should we put into row  $n$  and column  $-k$  of Pascal's Rectangle in order for the Pascal Equation to hold true?
  - \* What should we put into row  $-n$  (assume  $n$  is positive) and column  $k$  or column  $-k$  in order for the Pascal Equation to continue to hold? Do we have any freedom of choice?

32. There is yet another bijection that lets us prove that a set of size  $n$  has  $2^n$  subsets. Namely, for each subset  $S$  of  $[n] = \{1, 2, \dots, n\}$ , define a function (traditionally denoted by  $\chi_S$ ) as follows.<sup>1</sup>

$$\chi_S(i) = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

The function  $\chi_S$  is called the *characteristic function* of  $S$ . Notice that the characteristic function is a function from  $[n]$  to  $\{0, 1\}$ .

---

<sup>1</sup>The symbol  $\chi$  is the Greek letter chi that is pronounced Ki, with the  $i$  sounding like "eye."

- (a) For practice, consider the function  $\chi_{\{1,3\}}$  for the subset  $\{1, 3\}$  of the set  $\{1, 2, 3, 4\}$ . What are
- i.  $\chi_{\{1,3\}}(1)$ ?
  - ii.  $\chi_{\{1,3\}}(2)$ ?
  - iii.  $\chi_{\{1,3\}}(3)$ ?
  - iv.  $\chi_{\{1,3\}}(4)$ ?
- (b) We define a function  $f$  from the set of subsets of  $[n] = \{1, 2, \dots, n\}$  to the set of functions from  $[n]$  to  $\{0, 1\}$  by  $f(S) = \chi_S$ . Explain why  $f$  is a bijection.
- (c) Why does the fact that  $f$  is a bijection prove that  $[n]$  has  $2^n$  subsets?

In Problems 18, 27, and 32 you gave three proofs of the following theorem.

**Theorem 1** *The number of subsets of an  $n$ -element set is  $2^n$ .*

The proofs in Problem 27 and 32 use essentially the same bijection, but they interpret sequences of zeros and ones differently, and so end up being different proofs. We will give yet another proof, using bijections similar to those we used in proving the Pascal Equation, at the beginning of Chapter 2.

### 1.2.6 The quotient principle

•33. As we noted in Problem 28, the first question in Problem 8 asked us for the number of three-element subsets of a twelve-element set. We were able to use the Pascal Equation to get a numerical answer to that question. Had we had twenty or thirty flavors of ice cream to choose from, using the Pascal Equation to get our answer would have entailed a good bit more work. We have seen how the general product principle gives us an answer to Problem 6. Thus we might think that the number of ways to choose a three element set from 12 elements is the number of ways to choose the first element times the number of ways to choose the second element times the number of ways to choose the third element, which is  $12 \cdot 11 \cdot 10 = 1320$ . However, our result in Problem 28 shows that this is wrong.

- (a) What is it that is different between the number of ways to stack ice cream in a triple decker cone with three different flavors of ice cream and the number of ways to simply choose three different flavors of ice cream?

- (b) In particular, how many different triple decker cones use vanilla, chocolate, and strawberry? (Of course any three distinct flavors could substitute for vanilla, chocolate and strawberry without changing the answer.)
- (c) Using your answer from part 33b, compute the number of ways to choose three different flavors of ice cream (out of twelve flavors) from the number of ways to choose a triple decker cone with three different flavors (out of twelve flavors).
- 34. Based on what you observed in Problem 33c, how many  $k$ -element subsets does an  $n$ -element set have?
- 35. The formula you proved in Problem 34 is symmetric in  $k$  and  $n-k$ ; that is, it gives the same number for  $\binom{n}{k}$  as it gives for  $\binom{n}{n-k}$ . Whenever two quantities are counted by the same formula it is good for our insight to find a bijection that demonstrates the two sets being counted have the same size. In fact this is a guiding principle of research in combinatorial mathematics. Find a bijection that proves that  $\binom{n}{k}$  equals  $\binom{n}{n-k}$ .
- 36. In how many ways can we pass out  $k$  (identical) ping-pong balls to  $n$  children if each child may get at most one?
- 37. In how many ways may  $n$  people sit around a round table? (Assume that when people are sitting around a round table, all that really matters is who is to each person's right. For example, if we can get one arrangement of people around the table from another by having everyone get up and move to the right one place and sit back down, then we get an equivalent arrangement of people. Notice that you can get a list from a seating arrangement by marking a place at the table, and then listing the people at the table, starting at that place and moving around to the right.) There are at least two different ways of doing this problem. Try to find them both.

We are now going to analyze the result of Problem 34 in more detail in order to tease out another counting principle that we can use in a wide variety of situations.

In Table 1.2 we list all three-element permutations of the 5-element set  $\{a, b, c, d, e\}$ . Each row consists of all 3-element permutations of some subset of  $\{a, b, c, d, e\}$ . Because a given  $k$ -element subset can be listed as a  $k$ -element permutation in  $k!$  ways, there are  $3! = 6$  permutations in each row. Because each 3-element permutation appears exactly once in the table, each row is

Table 1.2: The 3-element permutations of  $\{a, b, c, d, e\}$  organized by which 3-element set they permute.

<i>abc</i>	<i>acb</i>	<i>bac</i>	<i>bca</i>	<i>cab</i>	<i>cba</i>
<i>abd</i>	<i>adb</i>	<i>bad</i>	<i>bda</i>	<i>dab</i>	<i>dba</i>
<i>abe</i>	<i>aeb</i>	<i>bae</i>	<i>bea</i>	<i>eab</i>	<i>eba</i>
<i>acd</i>	<i>adc</i>	<i>cad</i>	<i>cda</i>	<i>dac</i>	<i>dca</i>
<i>ace</i>	<i>aec</i>	<i>cae</i>	<i>cea</i>	<i>eac</i>	<i>eca</i>
<i>ade</i>	<i>aed</i>	<i>dae</i>	<i>dea</i>	<i>ead</i>	<i>eda</i>
<i>bcd</i>	<i>bdc</i>	<i>cbd</i>	<i>cdb</i>	<i>dbc</i>	<i>dcb</i>
<i>bce</i>	<i>bec</i>	<i>cbe</i>	<i>ceb</i>	<i>ebc</i>	<i>ecb</i>
<i>bde</i>	<i>bed</i>	<i>dbe</i>	<i>deb</i>	<i>ebd</i>	<i>edb</i>
<i>cde</i>	<i>ced</i>	<i>dce</i>	<i>dec</i>	<i>ecd</i>	<i>edc</i>

a block of a partition of the set of 3-element permutations of  $\{a, b, c, d, e\}$ . Each block has size six. Each block consists of all 3-element permutations of some three element subset of  $\{a, b, c, d, e\}$ . Since there are ten rows, we see that there are ten 3-element subsets of  $\{a, b, c, d, e\}$ . An alternate way to see this is to observe that we partitioned the set of all 60 three-element permutations of  $\{a, b, c, d, e\}$  into some number  $q$  of blocks, each of size six. Thus by the product principle,  $q \cdot 6 = 60$ , so  $q = 10$ .

- 38. Rather than restricting ourselves to  $n = 5$  and  $k = 3$ , we can partition the set of all  $k$ -element permutations of an  $n$ -element set  $S$  up into blocks by letting  $B_K$  be the set (block) of all  $k$ -element permutations of  $K$  for each  $k$ -element subset  $K$  of  $S$ . Thus as in our preceding example, each block consists of all permutations of some subset  $K$  of our  $n$ -element set. The questions that follow are about this partition of the set of  $k$ -element permutations of  $S$ .
  - (a) How many permutations are there in a block?
  - (b) Since  $S$  has  $n$  elements, what does Problem 20 tell you about the total number of  $k$ -element permutations of  $S$ ?
  - (c) Describe a bijection between the set of blocks of the partition and the set of  $k$ -element subsets of  $S$ .
  - (d) What formula does this give you for the number  $\binom{n}{k}$  of  $k$ -element subsets of an  $n$ -element set?

- 39. A basketball team has 12 players. However, only five players play at any given time during a game.
- (a) In how many ways may the coach choose the five players?
  - (b) To be more realistic, the five players playing a game normally consist of two guards, two forwards, and one center. If there are five guards, four forwards, and three centers on the team, in how many ways can the coach choose two guards, two forwards, and one center?
  - (c) What if one of the centers is equally skilled at playing forward?
- 40. In Problem 37, describe a way to partition the  $n$ -element permutations of the  $n$  people into blocks so that there is a bijection between the set of blocks of the partition and the set of arrangements of the  $n$  people around a round table. What method of solution for Problem 37 does this correspond to?
- 41. In Problems 38d and 40, you have been using the product principle in a new way. One of the ways in which we previously stated the product principle was “If we partition a set into  $m$  blocks each of size  $n$ , then the set has size  $m \cdot n$ .” In problems 38d and 40 we knew the size  $p$  of a set  $P$  of permutations of a set, and we knew we had partitioned  $P$  into some unknown number of blocks, each of a certain known size  $r$ . If we let  $q$  stand for the number of blocks, what does the product principle tell us about  $p$ ,  $q$ , and  $r$ ? What do we get when we solve for  $q$ ?

The formula you found in Problem 41 is so useful that we are going to single it out as another principle. The **quotient principle** says:

If we partition a set  $P$  of size  $p$  into  $q$  blocks, each of size  $r$ , then  $q = p/r$ .

The quotient principle is really just a restatement of the product principle, but thinking about it as a principle in its own right often leads us to find solutions to problems. Notice that it does not always give us a formula for the number of blocks of a partition; it only works when all the blocks have the same size. In Chapter 6, we develop a way to solve problems with different block sizes in cases where there is a good deal of symmetry in the problem. (The roundness of the table was a symmetry in the problem of people at a table; the fact that we can order the sets in any order is the symmetry in the problem of counting  $k$ -element subsets.)

In Section A.2 of Appendix A we introduce the idea of an equivalence relation, see what equivalence relations have to do with partitions, and discuss the quotient principle from that point of view. While that appendix is not required for what we are doing here, if you want a more thorough discussion of the quotient principle, this would be a good time to work through that appendix.

- 42. In how many ways may we string  $n$  distinct beads on a necklace without a clasp? (Perhaps we make the necklace by stringing the beads on a string, and then carefully gluing the two ends of the string together. Assume someone can pick up the necklace, move it around in space and put it back down, giving an apparently different way of stringing the beads that is equivalent to the first.)
- 43. We first gave this problem as Problem 12a. Now we have several ways to approach the problem. A tennis club has  $2n$  members. We want to pair up the members by twos for singles matches.
- In how many ways may we pair up all the members of the club? Give at least two solutions different from the one you gave in Problem 12a. (You may not have done Problem 12a. In that case, see if you can find three solutions.)
  - Suppose that in addition to specifying who plays whom, for each pairing we say who serves first. Now in how many ways may we specify our pairs? Try to find as many solutions as you can.
- 44. (This becomes especially relevant in Chapter 6, though it makes an important point here.) In how many ways may we attach two identical red beads and two identical blue beads to the corners of a square (with one bead per corner) free to move around in (three-dimensional) space?
- 45. While the formula you proved in Problem 34 and Problem 38d is very useful, it doesn't give us a sense of how big the binomial coefficients are. We can get a very rough idea, for example, of the size of  $\binom{2n}{n}$  by recognizing that we can write  $(2n)!/n!$  as  $\frac{2n}{n} \cdot \frac{2n-1}{n-1} \cdots \frac{n+1}{1}$ , and each quotient is at least 2, so the product is at least  $2^n$ . If this were an accurate estimate, it would mean the fraction of  $n$ -element subsets of a  $2n$ -element set would be about  $2^n/2^{2n} = 1/2^n$ , which becomes very small as  $n$  becomes large. However, it is pretty clear the approximation will not be a very good one, because some of the terms in that product



are much larger than 2. In fact, if  $\binom{2n}{k}$  were the same for every  $k$ , then each would be the fraction  $\frac{1}{2^{n+1}}$  of  $2^{2n}$ . This is much larger than the fraction  $\frac{1}{2^n}$ . But our intuition suggests that  $\binom{2n}{n}$  is much larger than  $\binom{2n}{1}$  and is likely larger than  $\binom{2n}{n-1}$  so we can be sure our approximation is a bad one. For estimates like this, James Stirling developed a formula to approximate  $n!$  when  $n$  is large, namely  $n!$  is about  $(\sqrt{2\pi n}) n^n / e^n$ . In fact the ratio of  $n!$  to this expression approaches 1 as  $n$  becomes infinite.<sup>2</sup> We write this as

$$n! \sim \sqrt{2\pi n} \frac{n^n}{e^n}.$$

We read this notation as  $n!$  is asymptotic to  $\sqrt{2\pi n} \frac{n^n}{e^n}$ . Use Stirling's formula to show that the fraction of subsets of size  $n$  in an  $2n$ -element set is approximately  $1/\sqrt{\pi n}$ . This is a much bigger fraction than  $\frac{1}{2^n}$ !

## 1.3 Some Applications of Basic Counting Principles

### 1.3.1 Lattice paths and Catalan Numbers

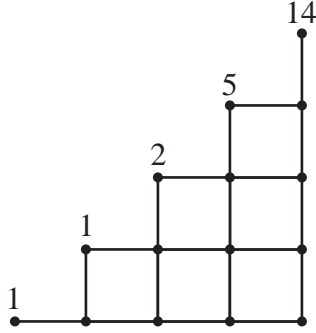
- 46. In a part of a city, all streets run either north-south or east-west, and there are no dead ends. Suppose we are standing on a street corner. In how many ways may we walk to a corner that is four blocks north and six blocks east, using as few blocks as possible?
- 47. Problem 46 has a geometric interpretation in a coordinate plane. A *lattice path* in the plane is a “curve” made up of line segments that either go from a point  $(i, j)$  to the point  $(i + 1, j)$  or from a point  $(i, j)$  to the point  $(i, j + 1)$ , where  $i$  and  $j$  are integers. (Thus lattice paths always move either up or to the right.) The length of the path is the number of such line segments.
- What is the length of a lattice path from  $(0, 0)$  to  $(m, n)$ ?
  - How many such lattice paths of that length are there?
  - How many lattice paths are there from  $(i, j)$  to  $(m, n)$ , assuming  $i, j, m,$  and  $n$  are integers?

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<sup>2</sup>Proving this takes more of a detour than is advisable here; however there is an elementary proof which you can work through in the problems of the end of Section 1 of Chapter 1 of *Introductory Combinatorics* by Kenneth P. Bogart, Harcourt Academic Press, (2000).

- 48. Another kind of geometric path in the plane is a *diagonal lattice path*. Such a path is a path made up of line segments that go from a point  $(i, j)$  to  $(i+1, j+1)$  (this is often called an *upstep*) or  $(i+1, j-1)$  (this is often called a *downstep*), again where  $i$  and  $j$  are integers. (Thus diagonal lattice paths always move towards the right but may move up or down.)
- Describe which points are connected to  $(0, 0)$  by diagonal lattice paths.
  - What is the length of a diagonal lattice path from  $(0, 0)$  to  $(m, n)$ ?
  - Assuming that  $(m, n)$  is such a point, how many diagonal lattice paths are there from  $(0, 0)$  to  $(m, n)$ ?
- 49. A school play requires a ten dollar donation per person; the donation goes into the student activity fund. Assume that each person who comes to the play pays with a ten dollar bill or a twenty dollar bill. The teacher who is collecting the money forgot to get change before the event. If there are always at least as many people who have paid with a ten as a twenty as they arrive the teacher won't have to give anyone an IOU for change. Suppose  $2n$  people come to the play, and exactly half of them pay with ten dollar bills.
- Describe a bijection between the set of sequences of tens and twenties people give the teacher and the set of lattice paths from  $(0, 0)$  to  $(n, n)$ .
  - Describe a bijection between the set of sequences of tens and twenties that people give the teacher and the set of diagonal lattice paths between  $(0, 0)$  and  $(2n, 0)$ .
  - In each case, what is the geometric interpretation of a sequence that does not require the teacher to give any IOUs?
- 50. Notice that a lattice path from  $(0, 0)$  to  $(n, n)$  stays inside (or on the edges of) the square whose sides are the  $x$ -axis, the  $y$ -axis, the line  $x = n$  and the line  $y = n$ . In this problem we will compute the number of lattice paths from  $(0, 0)$  to  $(n, n)$  that stay inside (or on the edges of) the triangle whose sides are the  $x$ -axis, the line  $x = n$  and the line  $y = x$ . For example, in Figure 1.7 we show the grid of points with integer coordinates for the triangle whose sides are the  $x$ -axis, the line  $x = 4$  and the line  $y = x$ .

Figure 1.7: The lattice paths from  $(0,0)$  to  $(i,i)$  for  $i = 0, 1, 2, 3, 4$ . The number of paths to the point  $(i,i)$  is shown just above that point.



- (a) Explain why the number of lattice paths from  $(0,0)$  to  $(n,n)$  that go outside the triangle described previously is the number of lattice paths from  $(0,0)$  to  $(n,n)$  that either touch or cross the line  $y = x + 1$ .
- (b) Find a bijection between lattice paths from  $(0,0)$  to  $(n,n)$  that touch (or cross) the line  $y = x + 1$  and lattice paths from  $(-1,1)$  to  $(n,n)$ .
- (c) Find a formula for the number of lattice paths from  $(0,0)$  to  $(n,n)$  that do not go above the line  $y = x$ . The number of such paths is called a *Catalan Number* and is usually denoted by  $C_n$ .
- 51. Your formula for the Catalan Number can be expressed as a binomial coefficient divided by an integer. Whenever we have a formula that calls for division by an integer, an ideal combinatorial explanation of the formula is one that uses the quotient principle. The purpose of this problem is to find such an explanation using diagonal lattice paths.<sup>3</sup> A diagonal lattice path that never goes below the  $y$ -coordinate of its first point is called a *Dyck Path*. We will call a Dyck Path from  $(0,0)$  to  $(2n,0)$  a *Catalan Path* of length  $2n$ . Thus the number of Catalan Paths of length  $2n$  is the Catalan Number  $C_n$ .

- (a) If a Dyck Path has  $n$  steps (each an upstep or downstep), why do

<sup>3</sup>The result we will derive is called the Chung-Feller Theorem; this approach is based on a paper of Wen-jin Woan “Uniform Partitions of Lattice Paths and Chung-Feller Generalizations,” **American Mathematics Monthly** 58 June/July 2001, p556.

the first  $k$  steps form a Dyck Path for each nonnegative  $k \leq n$ ?

- (b) Thought of as a curve in the plane, a diagonal lattice path can have many local maxima and minima, and can have several absolute maxima and minima, that is, several highest points and several lowest points. What is the  $y$ -coordinate of an absolute minimum point of a Dyck Path starting at  $(0, 0)$ ? Explain why a Dyck Path whose rightmost absolute minimum point is its last point is a Catalan Path.
- (c) Let  $D$  be the set of all diagonal lattice paths from  $(0, 0)$  to  $(2n, 0)$ . (Thus these paths can go below the  $x$ -axis.) Suppose we partition  $D$  by letting  $B_i$  be the set of lattice paths in  $D$  that have  $i$  upsteps (perhaps mixed with some downsteps) following the last absolute minimum. How many blocks does this partition have? Give a succinct description of the block  $B_0$ .
- (d) How many upsteps are in a Catalan Path?
- \* (e) We are going to give a bijection between the set of Catalan Paths and the block  $B_i$  for each  $i$  between 1 and  $n$ . For now, suppose the value of  $i$ , while unknown, is fixed. We take a Catalan path and break it into three pieces. The piece  $F$  (for “front”) consists of all steps before the  $i$ th upstep in the Catalan path. The piece  $U$  (for “up”) consists of the  $i$ th upstep. The piece  $B$  (for “back”) is the portion of the path that follows the  $i$ th upstep. Thus we can think of the path as  $FUB$ . Show that the function that takes  $FUB$  to  $BUF$  is a bijection from the set of Catalan Paths onto the block  $B_i$  of the partition. (Notice that  $BUF$  can go below the  $x$  axis.)
- (f) Explain how you have just given another proof of the formula for the Catalan Numbers.

### 1.3.2 The Binomial Theorem

- 52. We know that  $(x + y)^2 = x^2 + 2xy + y^2$ . Multiply both sides by  $(x + y)$  to get a formula for  $(x + y)^3$  and repeat to get a formula for  $(x + y)^4$ . Do you see a pattern? If so, what is it? If not, repeat the process to get a formula for  $(x + y)^5$  and look back at Figure 1.4 to see the pattern. Conjecture a formula for  $(x + y)^n$ .
- 53. When we apply the distributive law  $n$  times to  $(x + y)^n$ , we get a sum of terms of the form  $x^i y^{n-i}$  for various values of the integer  $i$ .

- (a) If it is clear to you that each term of the form  $x^i y^{n-i}$  that we get comes from choosing an  $x$  from  $i$  of the  $(x + y)$  factors and a  $y$  from the remaining  $n - i$  of the factors and multiplying these choices together, then answer this part of the problem and skip the next part. Otherwise, do the next part instead of this one. In how many ways can we choose an  $x$  from  $i$  terms and a  $y$  from  $n - i$  terms?

- i. Expand the product  $(x_1 + y_1)(x_2 + y_2)(x_3 + y_3)$ .
- ii. What do you get when you substitute  $x$  for each  $x_i$  and  $y$  for each  $y_i$ ?
- iii. Now imagine expanding

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n).$$

Once you apply the commutative law to the individual terms you get, you will have a sum of terms of the form

$$x_{k_1} x_{k_2} \cdots x_{k_i} \cdot y_{j_1} y_{j_2} \cdots y_{j_{n-i}}.$$

What is the set  $\{k_1, k_2, \dots, k_i\} \cup \{j_1, j_2, \dots, j_{n-i}\}$ ?

- iv. In how many ways can you choose the set  $\{k_1, k_2, \dots, k_i\}$ ?
- v. Once you have chosen this set, how many choices do you have for  $\{j_1, j_2, \dots, j_{n-i}\}$ ?
- vi. If you substitute  $x$  for each  $x_i$  and  $y$  for each  $y_i$ , how many terms of the form  $x^i y^{n-i}$  will you have in the expanded product

$$(x_1 + y_1)(x_2 + y_2) \cdots (x_n + y_n) = (x + y)^n?$$

- vii. How many terms of the form  $x^{n-i} y^i$  will you have?

- (b) Explain how you have just proved your conjecture from Problem 52. The theorem you have proved is called the **Binomial Theorem**.

54. What is  $\sum_{i=1}^{10} \binom{10}{i} 3^i$ ?

55. What is  $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n}$  if  $n$  is an integer bigger than zero?

→•56. Explain why

$$\sum_{i=0}^k \binom{m}{i} \binom{n}{k-i} = \binom{m+n}{k}.$$

Find two different explanations.

- 57. From the symmetry of the binomial coefficients, it is not too hard to see that when  $n$  is an odd number, the number of subsets of  $\{1, 2, \dots, n\}$  of odd size equals the number of subsets of  $\{1, 2, \dots, n\}$  of even size. Is it true that when  $n$  is even the number of subsets of  $\{1, 2, \dots, n\}$  of even size equals the number of subsets of odd size? Why or why not?
- 58. What is  $\sum_{i=0}^n i \binom{n}{i}$ ? (Hint: think about how you might use calculus.)

Notice how the proof you gave of the binomial theorem was a counting argument. It is interesting that an apparently algebraic theorem that tells us how to expand a power of a binomial is proved by an argument that amounts to counting the individual terms of the expansion. Part of the reason that combinatorial mathematics turns out to be so useful is that counting arguments often underlie important results of algebra. As the algebra becomes more sophisticated, so do the families of objects we have to count, but nonetheless we can develop a great deal of algebra on the basis of counting.

### 1.3.3 The pigeonhole principle

- 59. American coins are all marked with the year in which they were made. How many coins do you need to have in your hand to guarantee that on two (at least) of them, the date has the same last digit?

There are many ways in which you might explain your answer to Problem 59. For example, you can partition the coins according to the last digit of their date; that is, you put all the coins with a given last digit in a block together, and put no other coins in that block; repeating until all coins are in some block. Then you have a partition of your set of coins. If no two coins have the same last digit, then each block has exactly one coin. Since there are only ten digits, there are at most ten blocks and so by the sum principle there are at most ten coins. In fact with ten coins it is possible to have no two with the same last digit, but with 11 coins some block must have at least two coins in order for the sum of the sizes of at most ten blocks to be 11. This is one explanation of why we need 11 coins in Problem 59. This kind of situation arises often in combinatorial situations, and so rather than always using the sum principle to explain our reasoning, we enunciate another principle which we can think of as yet another variant of the sum principle. The **pigeonhole principle** states that

If we partition a set with more than  $n$  elements into  $n$  parts, then at least one part has more than one element.

The pigeonhole principle gets its name from the idea of a grid of little boxes that might be used, for example, to sort mail, or as mailboxes for a group of people in an office. The boxes in such grids are sometimes called pigeonholes in analogy with stacks of boxes used to house homing pigeons when homing pigeons were used to carry messages. People will sometimes state the principle in a more colorful way as “if we put more than  $n$  pigeons into  $n$  pigeonholes, then some pigeonhole has more than one pigeon.”

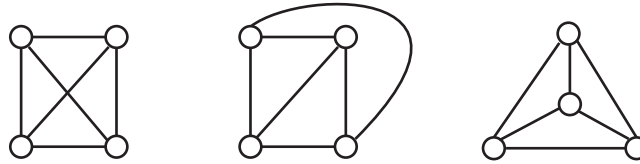
- 60. Show that if we have a function from a set of size  $n$  to a set of size less than  $n$ , then  $f$  is not one-to-one.
- 61. Show that if  $S$  and  $T$  are finite sets of the same size, then a function  $f$  from  $S$  to  $T$  is one-to-one if and only if it is onto.
- 62. There is a *generalized pigeonhole principle* which says that if we partition a set with more than  $kn$  elements into  $n$  blocks, then at least one block has at least  $k + 1$  elements. Prove the generalized pigeonhole principle.
- 63. All the powers of five end in a five, and all the powers of two are even. Show that for some integer  $n$ , if you take the first  $n$  powers of a prime other than two or five, one must have “01” as the last two digits.
- 64. Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (We assume that if person 1 knows person 2, then person 2 knows person 1.)
- 65. Draw five circles labeled Al, Sue, Don, Pam, and Jo. Find a way to draw red and green lines between people so that every pair of people is joined by a line and there is neither a triangle consisting entirely of red lines or a triangle consisting of green lines. What does Problem 64 tell you about the possibility of doing this with six people’s names? What does this problem say about the conclusion of Problem 64 holding when there are five people in our set rather than six?

#### 1.3.4 Ramsey Numbers

Problems 64 and 65 together show that six is the smallest number  $R$  with the property that if we have  $R$  people in a room, then there is either a set of (at least) three mutual acquaintances or a set of (at least) three mutual strangers. Another way to say the same thing is to say that six is the smallest

number so that no matter how we connect six points in the plane (no three on a line) with red and green lines, we can find either a red triangle or a green triangle. There is a name for this property. The **Ramsey Number**  $R(m, n)$  is the smallest number  $R$  so that if we have  $R$  people in a room, then there is a set of at least  $m$  mutual acquaintances or at least  $n$  mutual strangers. There is also a geometric description of Ramsey Numbers; it uses the idea of a *complete graph* on  $R$  vertices. A **complete graph** on  $R$  vertices consists of  $R$  points in the plane, together with line segments (or curves) connecting each two of the  $R$  vertices.<sup>4</sup> The points are called *vertices* and the line segments are called *edges*. In Figure 1.8 we show three different ways to draw a complete graph on four vertices. We use  $K_n$  to stand for a complete graph on  $n$  vertices.

Figure 1.8: Three ways to draw a complete graph on four vertices



Our geometric description of  $R(3, 3)$  may be translated into the language of graph theory (which is the subject that includes complete graphs) by saying  $R(3, 3)$  is the smallest number  $R$  so that if we color the edges of a  $K_R$  with two colors, then we can find in our picture a  $K_3$  all of whose edges have the same color. The graph theory description of  $R(m, n)$  is that  $R(m, n)$  is the smallest number  $R$  so that if we color the edges of a  $K_R$  with red and green, then we can find in our picture either a  $K_m$  all of whose edges are red or a  $K_n$  all of whose edges are green. Because we could have said our colors in the opposite order, we may conclude that  $R(m, n) = R(n, m)$ . In particular  $R(n, n)$  is the smallest number  $R$  such that if we color the edges of a  $K_R$  with two colors, then our picture contains a  $K_n$  all of whose edges have the same color.

- 66. Since  $R(3, 3) = 6$ , an uneducated guess might be that  $R(4, 4) = 8$ . Show that this is not the case.

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<sup>4</sup>As you may have guessed, a complete graph is a special case of something called a graph. The word graph will be defined in Section 2.3.1.



- 67. Show that among ten people, there are either four mutual acquaintances or three mutual strangers. What does this say about  $R(4, 3)$ ?
- 68. Show that among an odd number of people there is at least one person who is an acquaintance of an even number of people and therefore also a stranger to an even number of people.
- 69. Find a way to color the edges of a  $K_8$  with red and green so that there is no red  $K_4$  and no green  $K_3$ .
- 70. Find  $R(4, 3)$ .

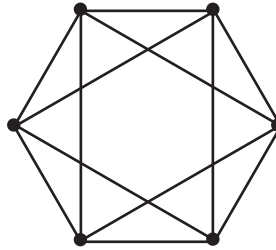
As of this writing, relatively few Ramsey Numbers are known.  $R(3, n)$  is known for  $n < 10$ ,  $R(4, 4) = 18$ , and  $R(5, 4) = R(4, 5) = 25$ .

## 1.4 Supplementary Chapter Problems

- 1. Remember that we can write  $n$  as a sum of  $n$  ones. How many plus signs do we use? In how many ways may we write  $n$  as a sum of a list of  $k$  positive numbers? Such a list is called a *composition* of  $n$  into  $k$  parts.
- 2. In Problem 1 we defined a composition of  $n$  into  $k$  parts. What is the total number of compositions of  $n$  (into any number of parts)?
- 3. Write down a list of all 16 zero-one sequences of length four starting with 0000 in such a way that each entry differs from the previous one by changing just one digit. This is called a Gray Code. That is, a *Gray Code* for 0-1 sequences of length  $n$  is a list of the sequences so that each entry differs from the previous one in exactly one place. Can you describe how to get a Gray Code for 0-1 sequences of length five from the one you found for sequences of length 4? Can you describe how to prove that there is a Gray code for sequences of length  $n$ ?
- 4. Use the idea of a Gray Code from Problem 3 to prove bijectively that the number of even-sized subsets of an  $n$ -element set equals the number of odd-sized subsets of an  $n$ -element set.
- 5. A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example,  $((((())))())$  is balanced and  $((())$  and  $((()()))()$  are not. How many balanced lists of  $n$  left and  $n$  right parentheses are there?

- \*6. Suppose we plan to put six distinct computers in a network as shown in Figure 1.9. The lines show which computers can communicate directly with which others. Consider two ways of assigning computers to the nodes of the network different if there are two computers that communicate directly in one assignment and that don't communicate directly in the other. In how many different ways can we assign computers to the network?

Figure 1.9: A computer network.



- 7. In a circular ice cream dish we are going to put four scoops of ice cream of four distinct flavors chosen from among twelve flavors. Assuming we place four scoops of the same size as if they were at the corners of a square, and recognizing that moving the dish doesn't change the way in which we have put the ice cream into the dish, in how many ways may we choose the ice cream and put it into the dish?
- 8. In as many ways as you can, show that  $\binom{n}{k}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{k}$ .
- 9. A tennis club has  $4n$  members. To specify a doubles match, we choose two teams of two people. In how many ways may we arrange the members into doubles matches so that each player is in one doubles match? In how many ways may we do it if we specify in addition who serves first on each team?
10. A town has  $n$  streetlights running along the north side of Main Street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?
- \*11. We have  $n$  identical ping-pong balls. In how many ways may we paint them red, white, blue, and green?

- \*12. We have  $n$  identical ping-pong balls. In how many ways may we paint them red, white, blue, and green if we use green paint on an even number of them?



## Chapter 2

# Applications of Induction and Recursion in Combinatorics and Graph Theory

### 2.1 Some Examples of Mathematical Induction

If you are unfamiliar with the Principle of Mathematical Induction, you should read Appendix B (a portion of which is repeated here).

#### 2.1.1 Mathematical induction

The **principle of mathematical induction** states that

In order to prove a statement about an integer  $n$ , if we can

1. Prove the statement when  $n = b$ , for some fixed integer  $b$ ,  
and
2. Show that the truth of the statement for  $n = k - 1$  implies  
the truth of the statement for  $n = k$  whenever  $k > b$ ,

then we can conclude the statement is true for all integers  $n \geq b$ .

As an example, let us give yet another proof that a set with  $n$  elements has  $2^n$  subsets. This proof uses essentially the the same bijections we used in proving the Pascal Equation. The statement we wish to prove is the statement that “A set of size  $n$  has  $2^n$  subsets.”

Our statement is true when  $n = 0$ , because a set of size 0 is the empty set and the empty set has  $1 = 2^0$  subsets. (This step of our proof is called a *base step*.)

Now suppose that  $k > 0$  and every set with  $k - 1$  elements has  $2^{k-1}$  subsets. Suppose  $S = \{a_1, a_2, \dots, a_k\}$  is a set with  $k$  elements. We partition the subsets of  $S$  into two blocks. Block  $B_1$  consists of the subsets that do not contain  $a_n$  and block  $B_2$  consists of the subsets that do contain  $a_n$ . Each set in  $B_1$  is a subset of  $\{a_1, a_2, \dots, a_{k-1}\}$ , and each subset of  $\{a_1, a_2, \dots, a_{k-1}\}$  is in  $B_1$ . Thus  $B_1$  is the set of all subsets of  $\{a_1, a_2, \dots, a_{k-1}\}$ . Therefore by our assumption in the first sentence of this paragraph, the size of  $B_1$  is  $2^{k-1}$ . Consider the function from  $B_2$  to  $B_1$  which takes a subset of  $S$  including  $a_k$  and removes  $a_k$  from it. This function is defined on  $B_2$ , because every set in  $B_2$  contains  $a_k$ . This function is onto, because if  $T$  is a set in  $B_1$ , then  $T \cup \{a_k\}$  is a set in  $B_2$  which the function sends to  $T$ . This function is one-to-one because if  $V$  and  $W$  are two different sets in  $B_2$ , then removing  $a_k$  from them gives two different sets in  $B_1$ . Thus we have a bijection between  $B_1$  and  $B_2$ , so  $B_1$  and  $B_2$  have the same size. Therefore by the sum principle the size of  $B_1 \cup B_2$  is  $2^{k-1} + 2^{k-1} = 2^k$ . Therefore  $S$  has  $2^k$  subsets. This shows that if a set of size  $k - 1$  has  $2^{k-1}$  subsets, then a set of size  $k$  has  $2^k$  subsets. Therefore by the principle of mathematical induction, a set of size  $n$  has  $2^n$  subsets for every nonnegative integer  $n$ .

The first sentence of the last paragraph is called the *inductive hypothesis*. In an inductive proof we always make an inductive hypothesis as part of proving that the truth of our statement when  $n = k - 1$  implies the truth of our statement when  $n = k$ . The last paragraph itself is called the *inductive step* of our proof. In an inductive step we derive the statement for  $n = k$  from the statement for  $n = k - 1$ , thus proving that the truth of our statement when  $n = k - 1$  implies the truth of our statement when  $n = k$ . The last sentence in the last paragraph is called the *inductive conclusion*. All inductive proofs should have a base step, an inductive hypothesis, an inductive step, and an inductive conclusion.

There are a couple details worth noticing. First, in this problem, our base step was the case  $n = 0$ , or in other words, we had  $b = 0$ . However, in other proofs,  $b$  could be any integer, positive, negative, or 0. Second, our proof that the truth of our statement for  $n = k - 1$  implies the truth of our statement for  $n = k$  required that  $k$  be at least 1, so that there would be an

element  $a_k$  we could take away in order to describe our bijection. However, condition (2) of the principle of mathematical induction only requires that we be able to prove the implication for  $k > 0$ , so we were allowed to assume  $k > 0$ .

### Strong Mathematical Induction

One way of looking at the principle of mathematical induction is that it tells us that if we know the “first” case of a theorem and we can derive each other case of the theorem from a smaller case, then the theorem is true in all cases. However, the particular way in which we stated the theorem is rather restrictive in that it requires us to derive each case from the immediately preceding case. This restriction is not necessary, and removing it leads us to a more general statement of the principal of mathematical induction which people often call the **strong principle of mathematical induction**. It states:

In order to prove a statement about an integer  $n$  if we can

1. Prove our statement when  $n = b$ , and
2. Prove that the statements we get with  $n = b, n = b + 1, \dots, n = k - 1$  imply the statement with  $n = k$ ,

then our statement is true for all integers  $n \geq b$ .

You will find some explicit examples of the use of the strong principle of mathematical induction in Appendix B and will find some uses for it in this chapter.

#### 2.1.2 Binomial Coefficients and the Binomial Theorem

- 71. When we studied the Pascal Equation and subsets in Chapter 1, it may have appeared that there is no connection between the Pascal relation  $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$  and the formula  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ . Of course you probably realize you can prove the Pascal relation by substituting the values the formula gives you into the right-hand side of the equation and simplifying to give you the left hand side. In fact, from the Pascal Relation and the facts that  $\binom{n}{0} = 1$  and  $\binom{n}{n} = 1$ , you can actually prove the formula for  $\binom{n}{k}$  by induction on  $n$ . Do so.
- 72. Use the fact that  $(x + y)^n = (x + y)(x + y)^{n-1}$  to give an inductive proof of the binomial theorem.

73. Suppose that  $f$  is a function defined on the nonnegative integers such that  $f(0) = 3$  and  $f(n) = 2f(n - 1)$ . Find a formula for  $f(n)$  and prove your formula is correct.
- + 74. Prove the conjecture in Problem 13b for an arbitrary positive integer  $m$  without appealing to the general product principle.

### 2.1.3 Inductive definition

You may have seen  $n!$  described by the two equations  $0! = 1$  and  $n! = n(n - 1)!$  for  $n > 0$ . By the principle of mathematical induction we know that this pair of equations defines  $n!$  for all nonnegative numbers  $n$ . For this reason we call such a definition an **inductive definition**. An inductive definition is sometimes called a *recursive definition*. Often we can get very easy proofs of useful facts by using inductive definitions.

- 75. An inductive definition of  $a^n$  for nonnegative  $n$  is given by  $a^0 = 1$  and  $a^n = aa^{n-1}$ . (Notice the similarity to the inductive definition of  $n!$ .) We remarked above that inductive definitions often give us easy proofs of useful facts. Here we apply this inductive definition to prove two useful facts about exponents that you have been using almost since you learned the meaning of exponents.
- (a) Use this definition to prove the rule of exponents  $a^{m+n} = a^m a^n$  for nonnegative  $m$  and  $n$ .
- (b) Use this definition to prove the rule of exponents  $a^{mn} = (a^m)^n$ .
- + 76. Suppose that  $f$  is a function on the nonnegative integers such that  $f(0) = 0$  and  $f(n) = n + f(n - 1)$ . Prove that  $f(n) = n(n + 1)/2$ . Notice that this gives a third proof that  $1 + 2 + \cdots + n = n(n + 1)/2$ , because this sum satisfies the two conditions for  $f$ . (The sum has no terms and is thus 0 when  $n = 0$ .)
- 77. Give an inductive definition of the summation notation  $\sum_{i=1}^n a_i$ . Use it and the distributive law  $b(a + c) = ba + bc$  to prove the distributive law

$$b \sum_{i=1}^n a_i = \sum_{i=1}^n ba_i.$$



### 2.1.4 Proving the general product principle (Optional)

We stated the sum principle as

If we have a partition of a set  $S$ , then the size of  $S$  is the sum of the sizes of the blocks of the partition.

In fact, the simplest form of the sum principle says that the size of the sum of two disjoint (finite) sets is the sum of their sizes.

78. Prove the sum principle we stated for partitions of a set from the simplest form of the sum principle.

We stated the simplest form of the product principle as

If we have a partition of a set  $S$  into  $m$  blocks, each of size  $n$ , then  $S$  has size  $mn$ .

In Problem 11 we gave a more general form of the product principle which can be stated as

Let  $S$  be a set of functions  $f$  from  $[n]$  to some set  $X$ . Suppose that

- there are  $k_1$  choices for  $f(1)$ , and
- for each choice of  $f(1), f(2), \dots, f(i-1)$ , there are  $k_i$  choices for  $f(i)$ .

Then the number of functions in the set  $S$  is  $k_1 k_2 \cdots k_n$ .

- + 79. Prove the general form of the product principle from the simplest form of the product principle.

### 2.1.5 Double Induction and Ramsey Numbers

In Section 1.3.4 we gave two different descriptions of the Ramsey number  $R(m, n)$ . However, if you look carefully, you will see that we never showed that Ramsey numbers actually exist; we merely described what they were and showed that  $R(3, 3)$  and  $R(3, 4)$  exist by computing them directly. As long as we can show that there is some number  $R$  such that when there are  $R$  people together, there are either  $m$  mutual acquaintances or  $n$  mutual strangers, this shows that the Ramsey Number  $R(m, n)$  exists, because it is the smallest such  $R$ . Mathematical induction allows us to show that one such  $R$  is  $\binom{m+n-2}{m-1}$ . The question is, what should we induct on,  $m$  or  $n$ ? In

other words, do we use the fact that with  $\binom{m+n-3}{m-2}$  people in a room there are at least  $m-1$  mutual acquaintances or  $n$  mutual strangers, or do we use the fact that with at least  $\binom{m+n-3}{n-2}$  people in a room there are at least  $m$  mutual acquaintances or at least  $n-1$  mutual strangers? It turns out that we use both. Thus we want to be able to simultaneously induct on  $m$  and  $n$ . One way to do that is to use yet another variation on the principle of mathematical induction, the *Principle of Double Mathematical Induction*. This principle (which can be derived from one of our earlier ones) states that

In order to prove a statement about integers  $m$  and  $n$ , if we can

1. Prove the statement when  $m = a$  and  $n = b$ , for fixed integers  $a$  and  $b$
2. Prove the statement when  $m = a$  and  $n > b$  and when  $m > a$  and  $n = b$  (for the same fixed integers  $a$  and  $b$ ),
3. Show that the truth of the statement for  $m = j$  and  $n = k-1$  (with  $j \geq a$  and  $k > j$ ) and the truth of the statement for  $m = j-1$  and  $n = k$  (with  $j > a$  and  $k \geq b$ ) imply the truth of the statement for  $m = j$  and  $n = k$ ,

then we can conclude the statement is true for all pairs of integers  $m \geq a$  and  $n \geq b$ .

- 80. Prove that  $R(m, n)$  exists by proving that if there are  $\binom{m+n-2}{m-1}$  people in a room, then there are either at least  $m$  mutual acquaintances or at least  $n$  mutual strangers.
81. Prove that  $R(m, n) \leq R(m-1, n) + R(m, n-1)$ .
- 82. (a) What does the equation in Problem 81 tell us about  $R(4, 4)$ ?
- \* (b) Consider 17 people arranged in a circle such that each person is acquainted with the first, second, fourth, and eighth person to the right and the first, second, fourth, and eighth person to the left. Can you find a set of four mutual acquaintances? Can you find a set of four mutual strangers?
- (c) What is  $R(4, 4)$ ?
83. (Optional) Prove the inequality of Problem 80 by induction on  $m+n$ .
84. Use Stirling's approximation (Problem 45) to convert the upper bound for  $R(n, n)$  that you get from Problem 80 to a multiple of a power of an integer.

### 2.1.6 A bit of asymptotic combinatorics

Problem 84 gives us an upper bound on  $R(n, n)$ . A very clever technique due to Paul Erdős, called the “probabilistic method,” will give a lower bound. Since both bounds are exponential in  $n$ , they show that  $R(n, n)$  grows exponentially as  $n$  gets large. An analysis of what happens to a function of  $n$  as  $n$  gets large is usually called an *asymptotic analysis*. The *probabilistic method*, at least in its simpler forms, can be expressed in terms of averages, so one does not need to know the language of probability in order to understand it. We will apply it to Ramsey numbers in the next problem. Combined with the result of Problem 84, this problem will give us that  $\sqrt{2}^n < R(n, n) < 2^{2n-2}$ , so that we know that the Ramsey number  $R(n, n)$  grows exponentially with  $n$ .

- 85. Suppose we have two numbers  $n$  and  $m$ . We consider all possible ways to color the edges of the complete graph  $K_m$  with two colors, say red and blue. For each coloring, we look at each  $n$ -element subset  $N$  of the vertex set  $M$  of  $K_m$ . Then  $N$  together with the edges of  $K_m$  connecting vertices in  $N$  forms a complete graph on  $n$  vertices. This graph, which we denote by  $K_N$ , has its edges colored by the original coloring of the edges of  $K_m$ .
- Why is it that if there is no subset  $N \subseteq M$  so that all the edges of  $K_N$  are colored the same color, then  $R(n, n) > m$ ?
  - To apply the probabilistic method, we are going to compute the average, over all colorings of  $K_m$ , of the number of sets  $N \subseteq M$  with  $|N| = n$  such that  $K_N$  *does* have all its edges the same color. Explain why it is that if the average is less than 1, then for some coloring there is no set  $N$  such that  $K_N$  has all its edges colored the same color. Why does this mean that  $R(n, n) > m$ ?
  - We call a  $K_N$  *monochromatic* for a coloring  $c$  of  $K_m$  if the color  $c(e)$  assigned to edge  $e$  is the same for every edge  $e$  of  $K_N$ . Let us define  $\text{mono}(c, N)$  to be 1 if  $N$  is monochromatic for  $c$  and to be 0 otherwise. Find a formula for the average number of monochromatic  $K_N$ s over all colorings of  $K_m$  that involves a double sum first over all edge colorings  $c$  of  $K_m$  and then over all  $n$ -element subsets  $N \subseteq M$  of  $\text{mono}(c, N)$ .
  - Show that your formula for the average reduces to  $2 \binom{m}{n} \cdot 2^{-\binom{n}{2}}$
  - Explain why  $R(n, n) > m$  if  $\binom{m}{n} \leq 2^{\binom{n}{2}-1}$ .

- \* (f) Explain why  $R(n, n) > \sqrt[n]{n!2^{\binom{n}{2}-1}}$ .
- (g) By using Stirling's formula, show that if  $n$  is large enough, then  $R(n, n) > \sqrt{2^n} = \sqrt{2}^n$ .

## 2.2 Recurrence Relations

86. How is the number of subsets of an  $n$ -element set related to the number of subsets of an  $(n - 1)$ -element set? Prove that you are correct.
87. Explain why it is that the number of bijections from an  $n$ -element set to an  $n$ -element set is equal to  $n$  times the number of bijections from an  $(n - 1)$ -element subset to an  $(n - 1)$ -element set. What does this have to do with Problem 26?

We can summarize these observations as follows. If  $s_n$  stands for the number of subsets of an  $n$ -element set, then

$$s_n = 2s_{n-1}, \quad (2.1)$$

and if  $b_n$  stands for the number of bijections from an  $n$ -element set to an  $n$ -element set, then

$$b_n = nb_{n-1}. \quad (2.2)$$

Equations 2.1 and 2.2 are examples of *recurrence equations* or *recurrence relations*. A **recurrence relation** or simply a **recurrence** is an equation that expresses the  $n$ th term of a sequence  $a_n$  in terms of values of  $a_i$  for  $i < n$ . Thus Equations 2.1 and 2.2 are examples of recurrences.

### 2.2.1 Examples of recurrence relations

Other examples of recurrences are

$$a_n = a_{n-1} + 7, \quad (2.3)$$

$$a_n = 3a_{n-1} + 2^n, \quad (2.4)$$

$$a_n = a_{n-1} + 3a_{n-2}, \text{ and} \quad (2.5)$$

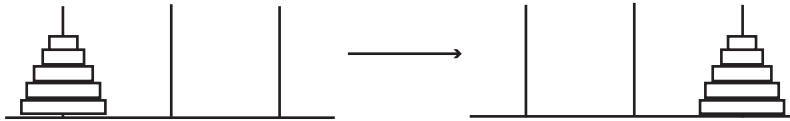
$$a_n = a_1a_{n-1} + a_2a_{n-2} + \cdots + a_{n-1}a_1. \quad (2.6)$$

A **solution** to a recurrence relation is a sequence that satisfies the recurrence relation. Thus a solution to Recurrence 2.1 is the sequence given by  $s_n = 2^n$ . Note that  $s_n = 17 \cdot 2^n$  and  $s_n = -13 \cdot 2^n$  are also solutions to Recurrence

2.1. What this shows is that a recurrence can have infinitely many solutions. In a given problem, there is generally one solution that is of interest to us. For example, if we are interested in the number of subsets of a set, then the solution to Recurrence 2.1 that we care about is  $s_n = 2^n$ . Notice this is the only solution we have mentioned that satisfies  $s_0 = 1$ .

88. Show that there is only one solution to Recurrence 2.1 that satisfies  $s_0 = 1$ .
89. A first-order recurrence relation is one which expresses  $a_n$  in terms of  $a_{n-1}$  and other functions of  $n$ , but which does not include any of the terms  $a_i$  for  $i < n - 1$  in the equation.
- (a) Which of the recurrences 2.1 through 2.6 are first order recurrences?
- (b) Show that there is one and only one sequence  $a_n$  that is defined for every nonnegative integer  $n$ , satisfies a given first order recurrence, and satisfies  $a_0 = a$  for some fixed constant  $a$ .

Figure 2.1: The Towers of Hanoi Puzzle



- 90. The “Towers of Hanoi” puzzle has three rods rising from a rectangular base with  $n$  rings of different sizes stacked in decreasing order of size on one rod. A legal move consists of moving a ring from one rod to another so that it does not land on top of a smaller ring. If  $m_n$  is the number of moves required to move all the rings from the initial rod to another rod that you choose, give a recurrence for  $m_n$ . (Hint: suppose you already knew the number of moves needed to solve the puzzle with  $n - 1$  rings.)
- 91. We draw  $n$  mutually intersecting circles in the plane so that each one crosses each other one exactly twice and no three intersect in the same point. (As examples, think of Venn diagrams with two or three mutually intersecting sets.) Find a recurrence for the number  $r_n$  of regions into which the plane is divided by  $n$  circles. (One circle

divides the plane into two regions, the inside and the outside.) Find the number of regions with  $n$  circles. For what values of  $n$  can you draw a Venn diagram showing all the possible intersections of  $n$  sets using circles to represent each of the sets?

### 2.2.2 Arithmetic Series (optional)

92. A child puts away two dollars from her allowance each week. If she starts with twenty dollars, give a recurrence for the amount  $a_n$  of money she has after  $n$  weeks and find out how much money she has at the end of  $n$  weeks.
93. A sequence that satisfies a recurrence of the form  $a_n = a_{n-1} + c$  is called an *arithmetic progression*. Find a formula in terms of the initial value  $a_0$  and the common difference  $c$  for the term  $a_n$  in an arithmetic progression and prove you are right.
94. A person who is earning \$50,000 per year gets a raise of \$3000 a year for  $n$  years in a row. Find a recurrence for the amount  $a_n$  of money the person earns over  $n + 1$  years. What is the total amount of money that the person earns over a period of  $n + 1$  years? (In  $n + 1$  years, there are  $n$  raises.)
95. An *arithmetic series* is a sequence  $s_n$  equal to the sum of the terms  $a_0$  through  $a_n$  of an arithmetic progression. Find a recurrence for the sum  $s_n$  of an arithmetic progression with initial value  $a_0$  and common difference  $c$  (using the language of Problem 93). Find a formula for general term  $s_n$  of an arithmetic series.

### 2.2.3 First order linear recurrences

Recurrences such as those in Equations 2.1 through 2.5 are called *linear recurrences*, as are the recurrences of Problems 90 and 91. A **linear recurrence** is one in which  $a_n$  is expressed as a sum of functions of  $n$  times values of (some of the terms)  $a_i$  for  $i < n$  plus (perhaps) another function (called the *driving function*) of  $n$ . A linear equation is called *homogeneous* if the driving function is zero (or, in other words, there is no driving function). It is called a *constant coefficient linear recurrence* if the functions that are multiplied by the  $a_i$  terms are all constants (but the driving function need not be constant).

96. Classify the recurrences in Equations 2.1 through 2.5 and Problems 90 and 91 according to whether or not they are constant coefficient, and whether or not they are homogeneous.
- 97. As you can see from Problem 96 some interesting sequences satisfy first order linear recurrences, including many that have constant coefficients, have constant driving term, or are homogeneous. Find a formula in terms of  $b$ ,  $d$ ,  $a_0$  and  $n$  for the general term  $a_n$  of a sequence that satisfies a constant coefficient first order linear recurrence  $a_n = ba_{n-1} + d$  and prove you are correct. If your formula involves a summation, try to replace the summation by a more compact expression.

### 2.2.4 Geometric Series

A sequence that satisfies a recurrence of the form  $a_n = ba_{n-1}$  is called a *geometric progression*. Thus the sequence satisfying Equation 2.1, the recurrence for the number of subsets of an  $n$ -element set, is an example of a geometric progression. From your solution to Problem 97, a geometric progression has the form  $a_n = a_0b^n$ . In your solution to Problem 97 you may have had to deal with the sum of a geometric progression in just slightly different notation, namely  $\sum_{i=0}^{n-1} db^i$ . A sum of this form is called a **(finite) geometric series**.

98. Do this problem only if your final answer (so far) to Problem 97 contained the sum  $\sum_{i=0}^{n-1} db^i$ .
- (a) Expand  $(1-x)(1+x)$ . Expand  $(1-x)(1+x+x^2)$ . Expand  $(1-x)(1+x+x^2+x^3)$ .
- (b) What do you expect  $(1-b)\sum_{i=0}^{n-1} db^i$  to be? What formula for  $\sum_{i=0}^{n-1} db^i$  does this give you? Prove that you are correct.

In Problem 97 and perhaps 98 you proved an important theorem. While the theorem does not have a name, the formula it states is called the **sum of a finite geometric series**.

**Theorem 2** *If  $b \neq 1$  and  $a_n = ba_{n-1} + d$ , then  $a_n = a_0b^n + d\frac{1-b^n}{1-b}$ . If  $b = 1$ , then  $a_n = a_0 + nd$ .*

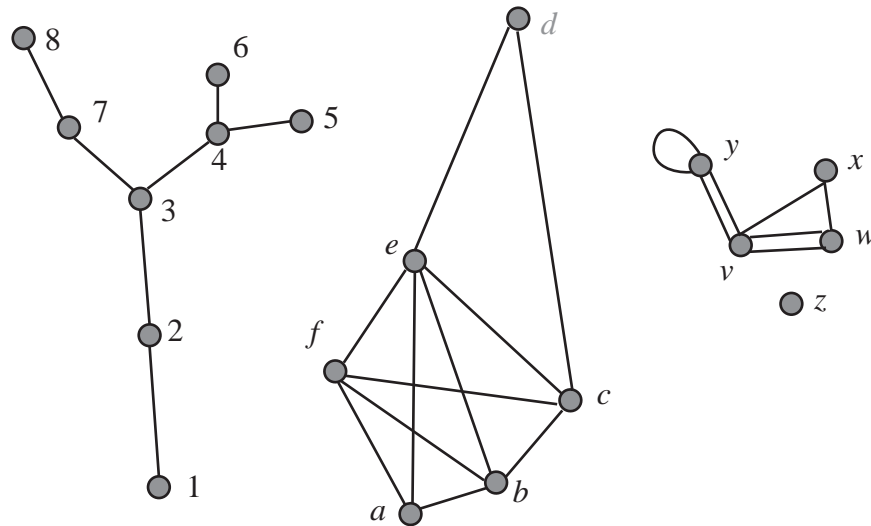
**Corollary 1** *If  $b \neq 1$ , then  $\sum_{i=0}^{n-1} b^i = \frac{1-b^n}{1-b}$ . If  $b = 1$ ,  $\sum_{i=0}^{n-1} b^i = n$ .*

## 2.3 Graphs and Trees

### 2.3.1 Undirected graphs

In Section 1.3.4 we introduced the idea of a directed graph. Graphs consist of vertices and edges. We describe vertices and edges in much the same way as we describe points and lines in geometry: we don't really say what vertices and edges are, but we say what they do. We just don't have a complicated axiom system the way we do in geometry. A **graph** consists of a set  $V$  called a vertex set and a set  $E$  called an edge set. Each member of  $V$  is called a *vertex* and each member of  $E$  is called an *edge*. Associated with each edge are two (not necessarily different) vertices called its endpoints. We draw pictures of graphs by drawing points to represent the vertices and line segments (curved if we choose) whose endpoints are at vertices to represent the edges. In Figure 2.2 we show three pictures of graphs. Each grey circle

Figure 2.2: Three different graphs



in the figure represents a vertex; each line segment represents an edge. You will note that we labelled the vertices; these labels are names we chose to give the vertices. We can choose names or not as we please. The third graph also shows that it is possible to have an edge that connects a vertex (like the one labelled  $y$ ) to itself or it is possible to have two or more edges (like those between vertices  $v$  and  $w$ ) between two vertices. The *degree* of a vertex



is the number of times it appears as the endpoint of edges; thus the degree of  $y$  in the third graph in the figure is four.

- 99. In the graph on the left in Figure 2.2, what is the degree of each vertex?
- 100. For each graph in Figure 2.2 is the number of vertices of odd degree even or odd?
- ·101. The sum of the degrees of the vertices of a (finite) graph is related in a natural way to the number of edges.
  - (a) What is the relationship?
  - (b) Find a proof that what you say is correct that uses induction on the number of edges. Hint: To make your inductive step, think about what happens to a graph if you delete an edge.
  - (c) Find a proof that what you say is correct that uses induction on the number of vertices.
  - (d) Find a proof that what you say is correct that does not use induction.
- 102. What can you say about the number of vertices of odd degree in a graph?

### 2.3.2 Walks and paths in graphs

A *walk* in a graph is an alternating sequence  $v_0e_1v_1 \dots e_iv_i$  of vertices and edges such that edge  $e_i$  connects vertices  $v_{i-1}$  and  $v_i$ . A graph is called connected if, for any pair of vertices, there is a walk starting at one and ending at the other.

- 103. Which of the graphs in Figure 2.2 is connected?
- 104. A *path* in a graph is a walk with no repeated vertices. Find the longest path you can in the third graph of Figure 2.2.
- 105. A *cycle* in a graph is a walk whose first and last vertex are equal but which has no other repeated vertices. Which graphs in Figure 2.2 have cycles? What is the largest number of edges in a cycle in the second graph in Figure 2.2? What is the smallest number of edges in a cycle in the third graph in Figure 2.2?
- 106. A connected graph with no cycles is called a **tree**. Which graphs, if any, in Figure 2.2 are trees?

### 2.3.3 Counting vertices, edges, and paths in trees

- · 107. Draw some trees and on the basis of your examples, make a conjecture about the relationship between the number of vertices and edges in a tree. Prove your conjecture. (Hint: what happens if you choose an edge and delete it, but not its endpoints?)
- 108. What is the minimum number of vertices of degree one in a finite tree? What is it if the number of vertices is bigger than one? Prove that you are correct.
- · 109. In a tree on any number of vertices, given two vertices, how many paths can you find between them? Prove that you are correct.
- 110. How many trees are there on the vertex set  $\{1, 2\}$ ? On the vertex set  $\{1, 2, 3\}$ ? When we label the vertices of our tree, we consider the tree which has edges between vertices 1 and 2 and between vertices 2 and 3 different from the tree that has edges between vertices 1 and 3 and between 2 and 3. See Figure 2.3. How many (labelled) trees are

Figure 2.3: The three labelled trees on three vertices



there on four vertices? How many (labelled) trees are there with five vertices? You don't have a lot of data to guess from, but try to guess a formula for the number of labelled trees with vertex set  $\{1, 2, \dots, n\}$ .

We are now going to introduce a method to prove the formula you guessed. Given a tree with two or more vertices, labelled with positive integers, we define a sequence  $b_1, b_2, \dots$  of integers inductively as follows: If the tree has two vertices, the sequence consists of one entry, namely the label of the vertex with the larger label. Otherwise, let  $a_1$  be the lowest numbered vertex of degree 1 in the tree. Let  $b_1$  be the label of the unique vertex in the tree adjacent to  $a_1$  and write down  $b_1$ . For example, in the first graph in Figure 2.2,  $a_1$  is 1 and  $b_1$  is 2. Given  $a_1$  through  $a_{i-1}$ , let  $a_i$  be the lowest numbered vertex of degree 1 in the tree you get by deleting  $a_1$  through  $a_{i-1}$  and let  $b_i$  be the unique vertex in this new tree adjacent to  $a_i$ . For example, in the first graph in Figure 2.2,  $a_2 = 2$  and  $b_2 = 3$ . Then

$a_3 = 5$  and  $b_3 = 4$ . We use  $b$  to stand for the sequence of  $b_i$ s we get in this way. In the tree (the first graph) in Figure 2.2, the sequence  $b$  is 2344378. (If you are unfamiliar with inductive (recursive) definition, you might want to write down some other labelled trees on eight vertices and construct the sequence of  $b_i$ s.)

111. (a) How long will the sequence of  $b_i$ s be if it is computed from a tree with  $n$  vertices (labelled with 1 through  $n$ )?  
 (b) What can you say about the last member of the sequence of  $b_i$ s?  
 (c) Can you tell from the sequence of  $b_i$ s what  $a_1$  is?  
 (d) Find a bijection between labelled trees and something you can “count” that will tell you how many labelled trees there are on  $n$  labelled vertices.

The sequence  $b_1, b_2, \dots, b_{n-2}$  in Problem 111 is called a *Prüfer coding* or *Prüfer code* for the tree. Thus the Prüfer code for the tree of Figure 2.2 is 234437. Notice that we do not include the term  $b_{n-1}$  in the Prüfer code because we know it is  $n$ . There is a good bit of interesting information encoded into the Prüfer code for a tree.

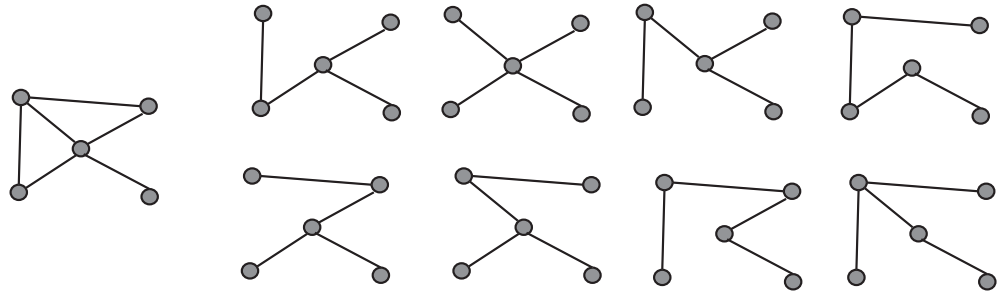
112. What can you say about the Prüfer code for a tree with exactly two vertices of degree 1 (and perhaps some vertices with other degrees as well)? Does this characterize such trees?
- 113. What can you determine about the degree of the vertex labelled  $i$  from the Prüfer code of the tree?
- 114. What is the number of (labelled) trees on  $n$  vertices with three vertices of degree 1? (Assume they are labelled with the integers 1 through  $n$ .) This problem will appear again in the next chapter after some material that will make it easier.

### 2.3.4 Spanning trees

Many of the applications of trees arise from trying to find an efficient way to connect all the vertices of a graph. For example, in a telephone network, at any given time we have a certain number of wires (or microwave channels, or cellular channels) available for use. These wires or channels go from a specific place to a specific place. Thus the wires or channels may be thought of as edges of a graph and the places the wires connect may be thought of as vertices of that graph. A tree whose edges are some of the edges of a

graph  $G$  and whose vertices are all of the vertices of the graph  $G$  is called a **spanning tree** of  $G$ . A spanning tree for a telephone network will give us a way to route calls between any two vertices in the network. In Figure 2.4 we show a graph and all its spanning trees.

Figure 2.4: A graph and all its spanning trees.



115. Show that every connected graph has a spanning tree. It is possible to find a proof that starts with the graph and works “down” towards the spanning tree and to find a proof that starts with just the vertices and works “up” towards the spanning tree. Can you find both kinds of proof?

### 2.3.5 Minimum cost spanning trees

Our motivation for talking about spanning trees was the idea of finding a minimum number of edges we need to connect all the edges of a communication network together. In many cases edges of a communication network come with costs associated with them. For example, one cell-phone operator charges another one when a customer of the first uses an antenna of the other. Suppose a company has offices in a number of cities and wants to put together a communication network connecting its various locations with high-speed computer communications, but to do so at minimum cost. Then it wants to take a graph whose vertices are the cities in which it has offices and whose edges represent possible communications lines between the cities. Of course there will not necessarily be lines between each pair of cities, and the company will not want to pay for a line connecting city  $i$  and city  $j$  if it can already connect them indirectly by using other lines it has chosen. Thus

it will want to choose a spanning tree of minimum cost among all spanning trees of the communications graph. For reasons of this application, if we have a graph with numbers assigned to its edges, the sum of the numbers on the edges of a spanning tree of  $G$  will be called the *cost* of the spanning tree.

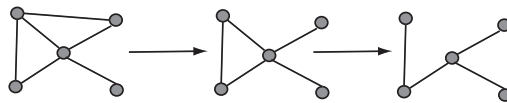
- 116. Describe a method (or better, two methods different in at least one aspect) for finding a spanning tree of minimum cost in a graph whose edges are labelled with costs, the cost on an edge being the cost for including that edge in a spanning tree. Prove that your method(s) work.

The method you used in Problem 116 is called a *greedy method*, because each time you made a choice of an edge, you chose the least costly edge available to you.

### 2.3.6 The deletion/contraction recurrence for spanning trees

There are two operations on graphs that we can apply to get a recurrence (though a more general kind than those we have studied for sequences) which will let us compute the number of spanning trees of a graph. The operations each apply to an edge  $e$  of a graph  $G$ . The first is called *deletion*; we *delete* the edge  $e$  from the graph by removing it from the edge set. Figure 2.5 shows how we can delete edges from a graph to get a spanning tree.

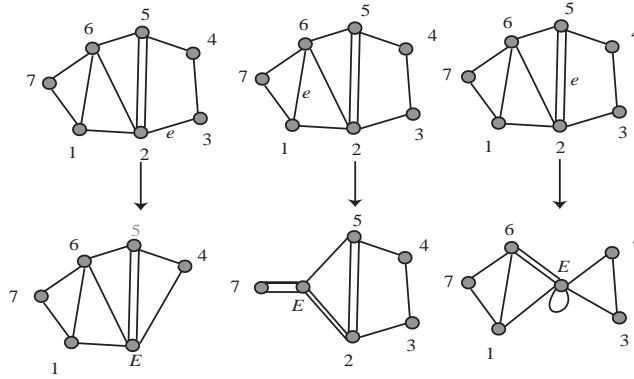
Figure 2.5: Deleting two appropriate edges from this graph gives a spanning tree.



The second operation is called *contraction*. Contractions of three different edges in the same graph are shown in Figure 2.6. Intuitively, we contract an edge by shrinking it in length until its endpoints coincide; we let the rest of the graph “go along for the ride.” To be more precise, we *contract* the edge  $e$  with endpoints  $v$  and  $w$  as follows:

1. remove all edges having either  $v$  or  $w$  or both as an endpoint from the edge set,

Figure 2.6: The results of contracting three different edges in a graph.

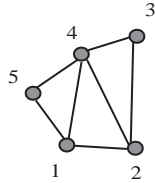


2. remove  $v$  and  $w$  from the vertex set,
3. add a new vertex  $E$  to the vertex set,
4. add an edge from  $E$  to each remaining vertex that used to be an endpoint of an edge whose other endpoint was  $v$  or  $w$ , and add an edge from  $E$  to  $E$  for any edge other than  $e$  whose endpoints were in the set  $\{v, w\}$ .

We use  $G - e$  (read as  $G$  minus  $e$ ) to stand for the result of deleting  $e$  from  $G$ , and we use  $G/e$  (read as  $G$  contract  $e$ ) to stand for the result of contracting  $e$  from  $G$ .

- ·117. How do the number of spanning trees of  $G$  not containing the edge  $e$  and the number of spanning trees of  $G$  containing  $e$  relate to the number of spanning trees of  $G - e$  and  $G/e$ ? Use  $\#(G)$  to stand for the number of spanning trees of  $G$  (so that, for example,  $\#(G/e)$  stands for the number of spanning trees of  $G/e$ ).
- (a) Find an expression for  $\#(G)$  in terms of  $\#(G/e)$  and  $\#(G - e)$ . This expression is called the *deletion-contraction recurrence*.
  - (b) Use the recurrence of the previous part to compute the number of spanning trees of the graph in Figure 2.7.

Figure 2.7: A graph.



### 2.3.7 Shortest paths in graphs

Suppose that a company has a main office in one city and regional offices in other cities. Most of the communication in the company is between the main office and the regional offices, so the company wants to find a spanning tree that minimizes not the total cost of all the edges, but rather the cost of communication between the main office and each of the regional offices. It is not clear that such a spanning tree even exists. This problem is a special case of the following. We have a connected graph with nonnegative numbers assigned to its edges. (In this situation these numbers are often called weights.) The (*weighted*) *length* of a path in the graph is the sum of the weights of its edges. The *distance* between two vertices is the least (weighted) length of any path between the two vertices. Given a vertex  $v$ , we would like to know the distance between  $v$  and each other vertex, and we would like to know if there is a spanning tree in  $G$  such that the length of the path in the spanning tree from  $v$  to each vertex  $x$  is the distance from  $v$  to  $x$  in  $G$ .

118. Show that the following algorithm (known as Dijkstra's algorithm) applied to a weighted graph whose vertices are labelled 1 to  $n$  gives, for each  $i$ , the distance from vertex 1 to  $i$  as  $d(i)$ .
- Let  $d(1) = 0$ . Let  $d(i) = \infty$  for all other  $i$ . Let  $v(1)=1$ . Let  $v(j) = 0$  for all other  $j$ . For each  $i$  and  $j$ , let  $w(i, j)$  be the minimum weight of an edge between  $i$  and  $j$ , or  $\infty$  if there are no such edges. Let  $k = 1$ . Let  $t = 1$ .
  - For each  $i$ , if  $d(i) > d(k) + w(k, i)$  let  $d(i) = d(k) + w(k, i)$ .
  - Among those  $i$  with  $v(i) = 0$ , choose one with  $d(i)$  a minimum, and let  $k = i$ . Increase  $t$  by 1. Let  $v(i) = 1$ .
  - Repeat the previous two steps until  $t = n$ .

119. Is there a spanning tree such that the distance from vertex 1 to vertex  $i$  given by the algorithm in Problem 118 is the distance from vertex 1 to vertex  $i$  in the tree (using the same weights on the edges, of course)?



## 2.4 Supplementary Problems

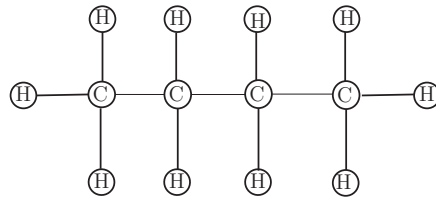
1. Use the inductive definition of  $a^n$  to prove that  $(ab)^n = a^n b^n$  for all nonnegative integers  $n$ .

2. Give an inductive definition of  $\bigcup_{i=1}^n S_i$  and use it and the two set distributive law to prove the distributive law

$$A \cap \bigcup_{i=1}^n S_i = \bigcup_{i=1}^n A \cap S_i.$$

- 3. A hydrocarbon molecule is a molecule whose only atoms are either carbon atoms or hydrogen atoms. In a simple molecular model of a hydrocarbon, a carbon atom will bond to exactly four other atoms and hydrogen atom will bond to exactly one other atom. Such a model is shown in Figure 2.8. We represent a hydrocarbon compound

Figure 2.8: A model of a butane molecule



with a graph whose vertices are labelled with C's and H's so that each C vertex has degree four and each H vertex has degree one. A hydrocarbon is called an "alkane" if the graph is a tree. Common examples are methane (natural gas), butane (one version of which is shown in Figure 2.8), propane, hexane (ordinary gasoline), octane (to make gasoline burn more slowly), etc.

- (a) How many vertices are labelled  $H$  in the graph of an alkane with exactly  $n$  vertices labelled  $C$ ?
  - (b) An alkane is called butane if it has four carbon atoms. Why do we say one version of butane is shown in Figure 2.8?
4. (a) Give a recurrence for the number of ways to divide  $2n$  people into sets of two for tennis games. (Don't worry about who serves first.)

- (b) Give a recurrence for the number of ways to divide  $2n$  people into sets of two for tennis games and to determine who serves first.)
- 5. Give a recurrence for the number of ways to divide  $4n$  people into sets of four for games of bridge. (Don't worry about how they sit around the bridge table or who is the first dealer.)
6. Use induction to prove your result in Supplementary Problem 2 at the end of Chapter 1.
7. Give an inductive definition of the product notation  $\prod_{i=1}^n a_i$ .
8. Using the fact that  $(ab)^k = a^k b^k$ , use your inductive definition of product notation in Problem 7 to prove that  $\left(\prod_{i=1}^n a_i\right)^k = \prod_{i=1}^n a_i^k$ .
- \*9. How many labelled trees on  $n$  vertices have exactly four vertices of degree 1? (This problem also appears in the next chapter since some ideas in that chapter make it more straightforward.)
- \*10. The *degree sequence* of a graph is a list of the degrees of the vertices in nonincreasing order. For example the degree sequence of the first graph in Figure 2.4 is  $(4, 3, 2, 2, 1)$ . For a graph with vertices labelled 1 through  $n$ , the *ordered degree sequence* of the graph is the sequence  $d_1, d_2, \dots, d_n$  in which  $d_i$  is the degree of vertex  $i$ . For example the ordered degree sequence of the first graph in Figure 2.2 is  $(1, 2, 3, 3, 1, 1, 2, 1)$ .
- (a) How many labelled trees are there on  $n$  vertices with ordered degree sequence  $d_1, d_2, \dots, d_n$ ? (This problem appears again in the next chapter since some ideas in that chapter make it more straightforward.)
- \* (b) How many labelled trees are there on  $n$  vertices with with the degree sequence in which the degree  $d$  appears  $i_d$  times?

## Chapter 3

# Distribution Problems

### 3.1 The Idea of a Distribution

Many of the problems we solved in Chapter 1 may be thought of as problems of distributing objects (such as pieces of fruit or ping-pong balls) to recipients (such as children). Some of the ways of viewing counting problems as distribution problems are somewhat indirect. For example, in Problem 36 you probably noticed that the number of ways to pass out  $k$  ping-pong balls to  $n$  children so that no child gets more than one is the number of ways that we may choose a  $k$ -element subset of an  $n$ -element set. We think of the children as recipients and objects we are distributing as the identical ping-pong balls, distributed so that each recipient gets at most one ball. Those children who receive an object are in our set. It is helpful to have more than one way to think of solutions to problems. In the case of distribution problems, another popular model for distributions is to think of putting balls in boxes rather than distributing objects to recipients. Passing out identical objects is modeled by putting identical balls into boxes. Passing out distinct objects is modeled by putting distinct balls into boxes.

#### 3.1.1 The twenty-fold way

When we are passing out objects to recipients, we may think of the objects as being either identical or distinct. We may also think of the recipients as being either identical (as in the case of putting fruit into plastic bags in the grocery store) or distinct (as in the case of passing fruit out to children). We may restrict the distributions to those that give at least one object to each recipient, or those that give exactly one object to each recipient, or those that give at most one object to each recipient, or we may have no

Table 3.1: An incomplete table of the number of ways to distribute  $k$  objects to  $n$  recipients, with restrictions on how the objects are received

The Twentyfold Way: A Table of Distribution Problems		
$k$ objects and conditions on how they are received	$n$ recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	$n^k$ functions	? set partitions ( $\leq n$ parts)
2. Distinct Each gets at most one	$n^{\underline{k}}$ $k$ -element permutations	1 if $k \leq n$ ; 0 otherwise
3. Distinct Each gets at least one	? onto functions	? set partitions ( $n$ parts)
4. Distinct Each gets exactly one	$k! = n!$ bijections	1 if $k = n$ ; 0 otherwise
5. Distinct, order matters	? ?	? ?
6. Distinct, order matters Each gets at least one	? ?	? ?
7. Identical no conditions	? ?	? ?
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$ ; 0 otherwise
9. Identical Each gets at least one	? ?	? ?
10. Identical Each gets exactly one	1 if $k = n$ ; 0 otherwise	1 if $k = n$ ; 0 otherwise

such restrictions. If the objects are distinct, it may be that the order in which the objects are received is relevant (think about putting books onto the shelves in a bookcase) or that the order in which the objects are received is irrelevant (think about dropping a handful of candy into a child's trick or treat bag). If we ignore the possibility that the order in which objects are received matters, we have created  $2 \cdot 2 \cdot 4 = 16$  distribution problems. In the cases where a recipient can receive more than one distinct object, we also have four more problems when the order objects are received matters. Thus we have 20 possible distribution problems.

We describe these problems in Table 3.1. Since there are twenty possible distribution problems, we call the table the "Twentyfold Way," adapting terminology suggested by Joel Spencer for a more restricted class of distribution problems. In the first column of the table we state whether the objects are distinct (like people) or identical (like ping-pong balls) and then give any conditions on how the objects may be received. The conditions we

consider are whether each recipient gets at most one object, whether each recipient gets at least one object, whether each recipient gets exactly one object, and whether the order in which the objects are received matters. In the second column we give the solution to the problem and the name of the mathematical model for this kind of distribution problem when the recipients are distinct, and in the third column we give the same information when the recipients are identical. We use question marks as the answers to problems we have not yet solved and models we have not yet studied. We give explicit answers to problems we solved in Chapter 1 and problems whose answers are immediate. The goal of this chapter is to develop methods that will allow us to fill in the table with formulas or at least quantities we know how to compute, and we will give a completed table at the end of the chapter. We will now justify the answers that are not question marks and replace some question marks with answers as we cover relevant material.

If we pass out  $k$  distinct objects (say pieces of fruit) to  $n$  distinct recipients (say children), we are saying for each object to which recipient it goes. Thus we are defining a function from the set of objects to the recipients. We saw the following theorem in Problem 13b.

**Theorem 3** *There are  $n^k$  functions from a  $k$ -element set to an  $n$ -element set.*

We proved it in one way in Problem 13b and in another way in Problem 74. If we pass out  $k$  distinct objects (say pieces of fruit) to  $n$  indistinguishable recipients (say identical paper bags) then we are dividing the objects up into disjoint sets; that is, we are forming a partition of the objects into some number, certainly no more than the number  $k$  of objects, of parts. Later in this chapter (and again in the next chapter) we shall discuss how to compute the number of partitions of a  $k$ -element set into  $n$  parts. This explains the entries in row one of our table.

If we pass out  $k$  distinct objects to  $n$  recipients so that each gets at most one, we still determine a function, but the function must be one-to-one. The number of one-to-one functions from a  $k$ -element set to an  $n$  element set is the same as the number of one-to-one functions from the set  $[k] = \{1, 2, \dots, k\}$  to an  $n$ -element set. In Problem 20 we proved the following theorem.

**Theorem 4** *If  $0 \leq k \leq n$ , then the number of  $k$ -element permutations of an  $n$ -element set is*

$$n^{\underline{k}} = n(n-1) \cdots (n-k+1) = n!/(n-k)!.$$

If  $k > n$  there are no one-to-one functions from a  $k$  element set to an  $n$  element set, so we define  $n^{\underline{k}}$  to be zero in this case. Notice that this is what the indicated product in the middle term of our formula gives us. If we are supposed to distribute  $k$  distinct objects to  $n$  identical recipients so that each gets at most one, we cannot do so if  $k > n$ , so there are 0 ways to do so. On the other hand, if  $k \leq n$ , then it doesn't matter which recipient gets which object, so there is only one way to do so. This explains the entries in row two of our table.

If we distribute  $k$  distinct objects to  $n$  distinct recipients so that each recipient gets at least one, then we are counting functions again, but this time functions from a  $k$ -element set *onto* an  $n$ -element set. At present we do not know how to compute the number of such functions, but we will discuss how to do so later in this chapter and in the next chapter. If we distribute  $k$  identical objects to  $n$  recipients, we are again simply partitioning the objects, but the condition that each recipient gets at least one means that we are partitioning the objects into exactly  $n$  blocks. Again, we will discuss how to compute the number of ways of partitioning a set of  $k$  objects into  $n$  blocks later in this chapter. This explains the entries in row three of our table.

If we pass out  $k$  distinct objects to  $n$  recipients so that each gets exactly one, then  $k = n$  and the function that our distribution gives us is a bijection. The number of bijections from an  $n$ -element set to an  $n$ -element set is  $n!$  by Theorem 4. If we pass out  $k$  distinct objects to  $n$  identical recipients so that each gets exactly 1, then in this case it doesn't matter which recipient gets which object, so the number of ways to do so is 1 if  $k = n$ . If  $k \neq n$ , then the number of such distributions is zero. This explains the entries in row four of our table.

We now jump to row eight of our table. We saw in Problem 36 that the number of ways to pass out  $k$  identical ping-pong balls to  $n$  children is simply the number of  $k$ -element subsets of an  $n$ -element set. In Problem 38d we proved the following theorem.

**Theorem 5** *If  $0 \leq k \leq n$ , the number of  $k$ -element subsets of an  $n$ -element set is given by*

$$\binom{n}{k} = \frac{n^{\underline{k}}}{k!} = \frac{n!}{k!(n-k)!}.$$

We define  $\binom{n}{k}$  to be 0 if  $k > n$ , because then there are no  $k$ -element subsets of an  $n$ -element set. Notice that this is what the middle term of the formula

in the theorem gives us. This explains the entries of row 8 of our table. For now we jump over row 9.

In row 10 of our table, if we are passing out  $k$  identical objects to  $n$  recipients so that each gets exactly one, it doesn't matter whether the recipients are identical or not; there is only one way to pass out the objects if  $k = n$  and otherwise it is impossible to make the distribution, so there are no ways of distributing the objects. This explains the entries of row 10 of our table. Several other rows of our table can be computed using the methods of Chapter 1.

### 3.1.2 Ordered functions

- 120. Suppose we wish to place  $k$  distinct books onto the shelves of a bookcase with  $n$  shelves. For simplicity, assume for now that all of the books would fit on any of the shelves. Also, let's imagine that once we are done putting books on the shelves, we push the books on a shelf as far to the left as we can, so that we are only thinking about how the books sit relative to each other, not about the exact places where we put the books. Since the books are distinct, we can think of the first book, the second book and so on. How many places are there where we can place the first book? When we place the second book, if we decide to place it on the shelf that already has a book, does it matter if we place it to the left or right of the book that is already there? How many places are there where we can place the second book? Once we have  $i - 1$  books placed, if we want to place book  $i$  on a shelf that already has some books, is sliding it in to the left of all the books already there different from placing it to the right of all the books already or between two books already there? In how many ways may we place the  $i$ th book into the bookcase? In how many ways may we place all the books?
121. Suppose we wish to place the books in Problem 120 (satisfying the assumptions we made there) so that each shelf gets at least one book. Now in how many ways may we place the books? (Hint: how can you make sure that each shelf gets at least one book before you start the process described in Problem 120?)

The assignment of which books go to which shelves of a bookcase is simply a function from the books to the shelves. But a function doesn't determine which book sits to the left of which others on the shelf, and this information is part of how the books are arranged on the shelves. In other words, the

order in which the shelves receive their books matters. Our function must thus assign an ordered list of books to each shelf. We will call such a function an ordered function. More precisely, an **ordered function** from a set  $S$  to a set  $T$  is a function that assigns an (ordered) list of elements of  $S$  to some, but not necessarily all, elements of  $T$  in such a way that each element of  $S$  appears on one and only one of the lists.<sup>1</sup> (Notice that although it is not the usual definition of a function from  $S$  to  $T$ , a function can be described as an assignment of subsets of  $S$  to some, but not necessarily all, elements of  $T$  so that each element of  $S$  is in one and only one of these subsets.) Thus the number of ways to place the books into the bookcase is the entry in the middle column of row 5 of our table. If in addition we require each shelf to get at least one book, we are discussing the entry in the middle column of row 6 of our table. An *ordered onto function* is one which assigns a list to each element of  $T$ .

In Problem 120 you showed that the number of ordered functions from a  $k$ -element set to an  $n$ -element set is  $\prod_{i=1}^k (n+i-1)$ . This product occurs frequently enough that it has a name; it is called the  *$k$ th rising factorial power* of  $n$  and is denoted by  $n^{\overline{k}}$ . It is read as “ $n$  to the  $k$  rising.” (This notation is due to Don Knuth, who also suggested the notation for falling factorial powers.) We can summarize with a theorem that adds two more formulas for the number of ordered functions.

**Theorem 6** *The number of ordered functions from a  $k$ -element set to an  $n$ -element set is*

$$n^{\overline{k}} = \prod_{i=1}^k (n+i-1) = \frac{(n+k-1)!}{(n-1)!} = (n+k-1)^{\underline{k}}.$$

Ordered Functions explain the entries in the middle column of rows 5 and 6 of our table of distribution problems.

### 3.1.3 Multisets

In the middle column of row 7 of our table, we are asking for the number of ways to distribute  $k$  identical objects (say ping-pong balls) to  $n$  distinct recipients (say children).

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<sup>1</sup>The phrase ordered function is not a standard one, because there is as yet no standard name for the result of an ordered distribution problem.



- 122. In how many ways may we distribute  $k$  identical books on the shelves of a bookcase with  $n$  shelves, assuming that any shelf can hold all the books?
- 123. A *multiset* chosen from a set  $S$  may be thought of as a subset with repeated elements allowed. To determine a multiset we must say how many times (including, perhaps, zero) each member of  $S$  appears in the multiset. The number of times an element appears is called its *multiplicity*. For example if we choose three identical red marbles, six identical blue marbles and four identical green marbles, from a bag of red, blue, green, white and yellow marbles then the multiplicity of a red marble in our multiset is three, while the multiplicity of a yellow marble is zero. The size of a multiset is sum of the multiplicities of its elements. For example if we choose three identical red marbles, six identical blue marbles and four identical green marbles, then the size of our multiset of marbles is 13. What is the number of multisets of size  $k$  that can be chosen from an  $n$ -element set?
- 124. Your answer in the previous problem should be expressible as a binomial coefficient. Since a binomial coefficient counts subsets, find a bijection between subsets of something and multisets chosen from a set  $S$ .
- 125. How many solutions are there in nonnegative integers to the equation  $x_1 + x_2 + \cdots + x_m = r$ , where  $m$  and  $r$  are constants?
- 126. In how many ways can we distribute  $k$  identical objects to  $n$  distinct recipients so that each recipient gets at least  $m$ ?

Multisets explain the entry in the middle column of row 7 of our table of distribution problems.

### 3.1.4 Compositions of integers

- 127. In how many ways may we put  $k$  identical books onto  $n$  shelves if each shelf must get at least one book?
- 128. A **composition** of the integer  $k$  into  $n$  parts is a list of  $n$  positive integers that add to  $k$ . How many compositions are there of an integer  $k$  into  $n$  parts?
- 129. Your answer in Problem 128 can be expressed as a binomial coefficient. This means it should be possible to interpret a composition as a subset

of some set. Find a bijection between compositions of  $k$  into  $n$  parts and certain subsets of some set. Explain explicitly how to get the composition from the subset and the subset from the composition.

- 130. Explain the connection between compositions of  $k$  into  $n$  parts and the problem of distributing  $k$  identical objects to  $n$  recipients so that each recipient gets at least one.

The sequence of problems you just completed should explain the entry in the middle column of row 9 of our table of distribution problems.

### 3.1.5 Broken permutations and Lah numbers

- ·131. In how many ways may we stack  $k$  distinct books into  $n$  identical boxes so that there is a stack in every box? (Hint: Imagine taking a stack of  $k$  books, and breaking it up into stacks to put into the boxes in the same order they were originally stacked. If you are going to use  $n$  boxes, in how many places will you have to break the stack up into smaller stacks, and how many ways can you do this?) (Alternate hint: How many different bookcase arrangements correspond to the same way of stacking  $k$  books into  $n$  boxes so that each box has at least one book?). The hints may suggest that you can do this problem in more than one way!

We can think of stacking books into identical boxes as partitioning the books and then ordering the blocks of the partition. This turns out not to be a useful computational way of visualizing the problem because the number of ways to order the books in the various stacks depends on the sizes of the stacks and not just the number of stacks. However, this way of thinking actually led to the first hint in Problem 131. Instead of dividing a set up into nonoverlapping parts, we may think of dividing a *permutation* (thought of as a list) of our  $k$  objects up into  $n$  ordered blocks. We will say that a set of ordered lists of elements of a set  $S$  is a **broken permutation** of  $S$  if each element of  $S$  is in one and only one of these lists.<sup>2</sup> The number of broken permutations of a  $k$ -element set with  $n$  blocks is denoted by  $L(k, n)$ . The number  $L(k, n)$  is called a *Lah Number* and, from our solution to Problem 131, is equal to  $k! \binom{k-1}{n-1} / n!$ .

The Lah numbers are the solution to the question “In how many ways may we distribute  $k$  distinct objects to  $n$  identical recipients if order matters

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<sup>2</sup>The phrase broken permutation is not standard, because there is no standard name for the solution to this kind of distribution problem.

and each recipient must get at least one?” Thus they give the entry in row 6 and column 6 of our table. The entry in row 5 and column 6 of our table will be the number of broken permutations with less than or equal to  $n$  parts. Thus it is a sum of Lah numbers.

We have seen that ordered functions and broken permutations explain the entries in rows 5 and 6 of our table.

The sequence of problems you just completed should explain the entry in the middle column of row 7 of our table of distribution problems. In the next two sections we will give ways of computing the remaining entries.

## 3.2 Partitions and Stirling Numbers

We have seen how the number of partitions of a set of  $k$  objects into  $n$  blocks corresponds to the distribution of  $k$  distinct objects to  $n$  identical recipients. While there is a formula that we shall eventually learn for this number, it requires more machinery than we now have available. However there is a good method for computing this number that is similar to Pascal’s equation. Now that we have studied recurrences in one variable, we will point out that Pascal’s equation is in fact a *recurrence in two variables*; that is, it lets us compute  $\binom{n}{k}$  in terms of values of  $\binom{m}{i}$  in which either  $m < n$  or  $i < k$  or both. It was the fact that we had such a recurrence and knew  $\binom{n}{0}$  and  $\binom{n}{n}$  that let us create Pascal’s triangle.

### 3.2.1 Stirling Numbers of the second kind

We use the notation  $S(k, n)$  to stand for the number of partitions of a  $k$  element set with  $n$  blocks. For historical reasons,  $S(k, n)$  is called a *Stirling Number of the second kind*.

- 132. In a partition of the set  $[k]$ , the number  $k$  is either in a block by itself, or it is not. How does the number of partitions of  $[k]$  with  $n$  parts in which  $k$  is in a block with other elements of  $[k]$  compare to the number of partitions of  $[k - 1]$  into  $n$  blocks? Find a two-variable recurrence for  $S(k, n)$ , valid for  $k$  and  $n$  larger than one.
133. What is  $S(k, 1)$ ? What is  $S(k, k)$ ? Create a table of values of  $S(k, n)$  for  $k$  between 1 and 5 and  $n$  between 1 and  $k$ . This table is sometimes called *Stirling’s Triangle (of the second kind)*. How would you define  $S(k, n)$  for the nonnegative values of  $k$  and  $n$  that are not both positive? Now for what values of  $k$  and  $n$  is your two variable recurrence valid?

134. Extend Stirling's triangle enough to allow you to answer the following question and answer it. (Don't fill in the rows all the way; the work becomes quite tedious if you do. Only fill in what you need to answer this question.) A caterer is preparing three bag lunches for hikers. The caterer has nine different sandwiches. In how many ways can these nine sandwiches be distributed into three identical lunch bags so that each bag gets at least one?
135. The question in Problem 134 naturally suggests a more realistic question; in how many ways may the caterer distribute the nine sandwiches into three identical bags so that each bag gets exactly three? Answer this question. (Hint, what if the question asked about six sandwiches and two distinct bags? How does having identical bags change the answer?)
- 136. What is  $S(k, k - 1)$ ?
- 137. In how many ways can we partition  $k$  items into  $n$  blocks so that we have  $k_i$  blocks of size  $i$  for each  $i$ ? (Notice that  $\sum_{i=1}^k k_i = n$  and  $\sum_{i=1}^k ik_i = k$ .) The sequence  $k_1, k_2, \dots, k_n$  is called the *type vector* of the partition.
138. Describe how to compute  $S(n, k)$  in terms of quantities given by the formula you found in Problem 137.
- 139. Find a recurrence for the Lah numbers  $L(k, n)$  similar to the one in Problem 132.
- 140. (Relevant in Appendix C.) The total number of partitions of a  $k$ -element set is denoted by  $B(k)$  and is called the  $k$ -th *Bell number*. Thus  $B(1) = 1$  and  $B(2) = 2$ .
- Show, by explicitly exhibiting the partitions, that  $B(3) = 5$ .
  - Find a recurrence that expresses  $B(k)$  in terms of  $B(n)$  for  $n < k$  and prove your formula correct in as many ways as you can.
  - Find  $B(k)$  for  $k = 4, 5, 6$ .

### 3.2.2 Stirling Numbers and onto functions

- 141. Given a function  $f$  from a  $k$ -element set  $K$  to an  $n$ -element set, we can define a partition of  $K$  by putting  $x$  and  $y$  in the same block of the partition if and only if  $f(x) = f(y)$ . How many blocks does the

partition have if  $f$  is onto? How is the number of functions from a  $k$ -element set onto an  $n$ -element set related to a Stirling number? Be as precise in your answer as you can.

- 142. How many labeled trees on  $n$  vertices have exactly 3 vertices of degree one? Note that this problem has appeared before in Chapter 2.
- 143. Each function from a  $k$ -element set  $K$  to an  $n$ -element set  $N$  is a function from  $K$  onto *some* subset of  $N$ . If  $J$  is a subset of  $N$  of size  $j$ , you know how to compute the number of functions that map onto  $J$  in terms of Stirling numbers. Suppose you add the number of functions mapping onto  $J$  over all possible subsets  $J$  of  $N$ . What simple value should this sum equal? Write the equation this gives you.
- 144. In how many ways can the sandwiches of Problem 134 be placed into three distinct bags so that each bag gets at least one?
- 145. In how many ways can the sandwiches of Problem 135 be placed into distinct bags so that each bag gets exactly three?
- 146. In how many ways may we label the elements of a  $k$  element set with  $n$  distinct labels (numbered 1 through  $n$ ) so that label  $i$  is used  $j_i$  times? If we think of the labels as  $y_1, y_2, \dots, y_n$ , then we can rephrase this question as follows. How many functions are there from a  $k$ -element set  $K$  to a set  $N = \{y_1, y_2, \dots, y_n\}$  so that each  $y_i$  is the image of  $j_i$  elements of  $K$ ? This number is called a *multinomial coefficient* and denoted by

$$\binom{k}{j_1, j_2, \dots, j_n}.$$

147. Explain how to compute the number of functions from a  $k$ -element set  $K$  to an  $n$ -element set  $N$  by using multinomial coefficients.
148. Explain how to compute the number of functions from a  $k$ -element set  $K$  onto an  $n$ -element set  $N$  by using multinomial coefficients.
- 149. What do multinomial coefficients have to do with expanding the  $k$ th power of a multinomial  $x_1 + x_2 + \dots + x_n$ ? This result is called the *multinomial theorem*.

### 3.2.3 Stirling Numbers and bases for polynomials

- 150. (a) Find a way to express  $n^k$  in terms of  $n^j$  for appropriate values  $j$ . You may use Stirling numbers if they help you.
- (b) Notice that  $x^j$  makes sense for a numerical variable  $x$  (that could range over the rational numbers, the real numbers, or even the complex numbers instead of only the nonnegative integers, as we are implicitly assuming  $n$  does), just as  $x^j$  does. Find a way to express the power  $x^k$  in terms of the polynomials  $x^j$  for appropriate values of  $j$  and explain why your formula is correct.

You showed in Problem 150b how to get each power of  $x$  in terms of the falling factorial powers  $x^{\underline{j}}$ . Therefore every polynomial in  $x$  is expressible in terms of a sum of numerical multiples of falling factorial powers. Using the language of linear algebra, we say that the ordinary powers of  $x$  and the falling factorial powers of  $x$  each form a basis for the “space” of polynomials, and that the numbers  $S(k, n)$  are “change of basis coefficients.” If you are not familiar with linear algebra, a *basis* for the *space of polynomials*<sup>3</sup> is a set of polynomials such that each polynomial, whether in that set or not, can be expressed in one and only one way as a sum of numerical multiples of polynomials in the set.

- 151. Show that every power of  $x + 1$  is expressible as a sum of numerical multiples of powers of  $x$ . Now show that every power of  $x$  (and thus every polynomial in  $x$ ) is a sum of numerical multiples (some of which could be negative) of powers of  $x + 1$ . This means that the powers of  $x + 1$  are a basis for the space of polynomials as well. Describe the change of basis coefficients that we use to express the binomial powers  $(x + 1)^n$  in terms of the ordinary  $x^j$  explicitly. Find the change of basis coefficients we use to express the ordinary powers  $x^n$  in terms of the binomial powers  $(x + 1)^k$ .
- • 152. By multiplication, we can see that every falling factorial polynomial can be expressed as a sum of numerical multiples of powers of  $x$ . In symbols, this means that there are numbers  $s(k, n)$  (notice that this  $s$  is lower case, not upper case) such that we may write  $x^{\underline{k}} = \sum_{n=0}^k s(k, n)x^n$ . These numbers  $s(k, n)$  are called Stirling Numbers of the first kind. By thinking algebraically about what the formula

$$x^{\underline{k}} = x^{\underline{k-1}}(x - k + 1) \quad (3.1)$$

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<sup>3</sup>The space of polynomials is just another name for the set of all polynomials.

means, we can find a recurrence for Stirling numbers of the first kind that gives us another triangular array of numbers called Stirling's triangle of the first kind. Explain why Equation 3.1 is true and use it to derive a recurrence for  $s(k, n)$  in terms of  $s(k-1, n-1)$  and  $s(k-1, n)$ .

153. Write down the rows of Stirling's triangle of the first kind for  $k = 0$  to 6.

By definition, the Stirling numbers of the first kind are also change of basis coefficients. The Stirling numbers of the first and second kind are change of basis coefficients from the falling factorial powers of  $x$  to the ordinary factorial powers, and vice versa.

- 154. Explain why every rising factorial polynomial  $x^{\overline{k}}$  can be expressed in terms of the falling factorial polynomials  $x^{\underline{n}}$ . Let  $b(k, n)$  stand for the change of basis coefficients that allow us to express  $x^{\overline{k}}$  in terms of the falling factorial polynomials  $x^{\underline{n}}$ ; that is, define  $b(k, n)$  by the equations

$$x^{\overline{k}} = \sum_{n=0}^k b(k, n)x^{\underline{n}}.$$

- Find a recurrence for  $b(k, n)$ .
- Find a formula for  $b(k, n)$  and prove the correctness of what you say in as many ways as you can.
- Is  $b(k, n)$  the same as any of the other families of numbers (binomial coefficients, Bell numbers, Stirling numbers, Lah numbers, etc.) we have studied?
- Say as much as you can (but say it precisely) about the change of basis coefficients for expressing  $x^{\overline{k}}$  in terms of  $x^{\overline{n}}$ .

### 3.3 Partitions of Integers

We have now completed all our distribution problems except for those in which both the objects and the recipients are identical. For example, we might be putting identical apples into identical paper bags. In this case all that matters is how many bags get one apple (how many recipients get one object), how many get two, how many get three, and so on. Thus for each bag we have a number, and the multiset of numbers of apples in the various bags is what determines our distribution of apples into identical bags. A multiset of positive integers that add to  $n$  is called a **partition** of  $n$ . Thus

the partitions of 3 are  $1+1+1$ ,  $1+2$  (which is the same as  $2+1$ ) and 3. The number of partitions of  $k$  is denoted by  $P(k)$ ; in computing the partitions of 3 we showed that  $P(3) = 3$ . It is traditional to use Greek letters like  $\lambda$  (the Greek letter  $\lambda$  is pronounced LAMB duh) to stand for partitions; we might write  $\lambda = 1, 1, 1$ ,  $\gamma = 2, 1$  and  $\tau = 3$  to stand for the three partitions we just described. We also write  $\lambda = 1^3$  as a shorthand for  $\lambda = 1, 1, 1$ , and we write  $\lambda \vdash 3$  as a shorthand for “ $\lambda$  is a partition of three.”

- 155. Find all partitions of 4 and find all partitions of 5, thereby computing  $P(4)$  and  $P(5)$ .

### 3.3.1 The number of partitions of $k$ into $n$ parts

A *partition of the integer  $k$  into  $n$  parts* is a multiset of  $n$  positive integers that add to  $k$ . We use  $P(k, n)$  to denote the number of partitions of  $k$  into  $n$  parts. Thus  $P(k, n)$  is the number of ways to distribute  $k$  identical objects to  $n$  identical recipients so that each gets at least one.

- 156. Find  $P(6, 3)$  by finding all partitions of 6 into 3 parts. What does this say about the number of ways to put six identical apples into three identical bags so that each bag has at least one apple?

### 3.3.2 Representations of partitions

- 157. How many solutions are there in the positive integers to the equation  $x_1 + x_2 + x_3 = 7$  with  $x_1 \geq x_2 \geq x_3$ ?
- 158. Explain the relationship between partitions of  $k$  into  $n$  parts and lists  $x_1, x_2, \dots, x_n$  of positive integers with  $x_1 \geq x_2 \geq \dots \geq x_n$ . Such a representation of a partition is called a *decreasing list* representation of the partition.
- 159. Describe the relationship between partitions of  $k$  and lists or vectors  $(x_1, x_2, \dots, x_n)$  such that  $x_1 + 2x_2 + \dots + nx_n = k$ . Such a representation of a partition is called a *type vector* representation of a partition, and it is typical to leave the trailing zeros out of such a representation; for example  $(2, 1)$  stands for the same partition as  $(2, 1, 0, 0)$ . What is the decreasing list representation for this partition, and what number does it partition?
- 160. How does the number of partitions of  $k$  relate to the number of partitions of  $k + 1$  whose smallest part is one?



When we write a partition as  $\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$ , it is customary to write the list of  $\lambda_i$ s as a decreasing list. When we have a type vector  $(t_1, t_2, \dots, t_m)$  for a partition, we write either  $\lambda = 1^{t_1} 2^{t_2} \dots m^{t_m}$  or  $\lambda = m^{t_m} (m-1)^{t_{m-1}} \dots 2^{t_2} 1^{t_1}$ . Henceforth we will use the second of these. When we write  $\lambda = \lambda_1^{i_1} \lambda_2^{i_2} \dots \lambda_n^{i_n}$ , we will assume that  $\lambda_i > \lambda_i + 1$ .

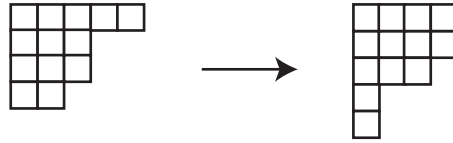
### 3.3.3 Ferrers and Young Diagrams and the conjugate of a partition

The decreasing list representation of partitions leads us to a handy way to visualize partitions. Given a decreasing list  $(\lambda_1, \lambda_2, \dots, \lambda_n)$ , we draw a figure made up of rows of dots that has  $\lambda_1$  equally spaced dots in the first row,  $\lambda_2$  equally spaced dots in the second row, starting out right below the beginning of the first row and so on. Equivalently, instead of dots, we may use identical squares, drawn so that a square touches each one to its immediate right or immediately below it along an edge. See Figure 3.1 for examples. The figure we draw with dots is called the Ferrers diagram of the partition; sometimes the figure with squares is also called a Ferrers diagram; sometimes it is called a Young diagram. At this stage it is irrelevant which name we choose and which kind of figure we draw; in more advanced work the squares are handy because we can put things like numbers or variables into them. From now on we will use squares and call the diagrams Young diagrams.

Figure 3.1: The Ferrers and Young diagrams of the partition (5,3,3,2)



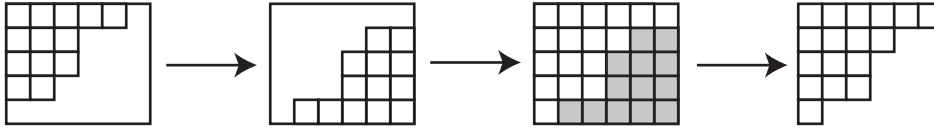
- 161. Draw the Young diagram of the partition (4,4,3,1,1). Describe the geometric relationship between the Young diagram of (5,3,3,2) and the Young diagram of (4,4,3,1,1).
- 162. The partition  $(\lambda_1, \lambda_2, \dots, \lambda_n)$  is called the *conjugate* of the partition  $(\gamma_1, \gamma_2, \dots, \gamma_m)$  if we obtain the Young diagram of one from the Young diagram of the other by flipping one around the line with slope -1 that extends the diagonal of the top left square. See Figure 3.2 for an example. What is the conjugate of (4,4,3,1,1)? How is the largest part

Figure 3.2: The Ferrers diagram the partition  $(5,3,3,2)$  and its conjugate.

of a partition related to the number of parts of its conjugate? What does this tell you about the number of partitions of a positive integer  $k$  with largest part  $m$ ?

- 163. A partition is called *self-conjugate* if it is equal to its conjugate. Find a relationship between the number of self-conjugate partitions of  $k$  and the number of partitions of  $k$  into distinct odd parts.
164. Explain the relationship between the number of partitions of  $k$  into even parts and the number of partitions of  $k$  into parts of even multiplicity, i.e. parts which are each used an even number of times as in  $(3,3,3,3,2,2,1,1)$ .
- 165. Show that the number of partitions of  $k$  into four parts equals the number of partitions of  $3k$  into four parts of size at most  $k - 1$  (or  $3k - 4$  into four parts of size at most  $k - 2$  or  $3k + 4$  into four parts of size at most  $k$ ).
166. The idea of conjugation of a partition could be defined without the geometric interpretation of a Young diagram, but it would seem far less natural without the geometric interpretation. Another idea that seems much more natural in a geometric context is this. Suppose we have a partition of  $k$  into  $n$  parts with largest part  $m$ . Then the Young diagram of the partition can fit into a rectangle that is  $m$  or more units wide (horizontally) and  $n$  or more units deep. Suppose we place the Young diagram of our partition in the top left-hand corner of an  $m'$  unit wide and  $n'$  unit deep rectangle with  $m' \geq m$  and  $n' \geq n$ , as in Figure 3.3. Why can we interpret the part of the rectangle not occupied by our Young diagram, rotated in the plane, as the Young diagram of another partition? This is called the *complement* of our partition in the rectangle. What integer is being partitioned by the complement? What conditions on  $m'$  and  $n'$  guarantee that the complement has the same number of parts as the original one? What

Figure 3.3: To complement the partition (5,3,3,2) in a 6 by 5 rectangle: enclose it in the rectangle, rotate, and cut out the original Young diagram.



conditions on  $m'$  and  $n'$  guarantee that the complement has the same largest part as the original one? Is it possible for the complement to have both the same number of parts and the same largest part as the original one? If we complement a partition in an  $m'$  by  $n'$  box and then complement that partition in an  $m'$  by  $n'$  box again, do we get the same partition that we started with?

- 167. Suppose we take a partition of  $k$  into  $n$  parts with largest part  $m$ , complement it in the smallest rectangle it will fit into, complement the result in the smallest rectangle it will fit into, and continue the process until we get the partition 1 of one into one part. What can you say about the partition with which we started?

168. Show that  $P(k, n)$  is at least  $\frac{1}{n!} \binom{k-1}{n-1}$ .

With the binomial coefficients, with Stirling numbers of the second kind, and with the Lah numbers, we were able to find a recurrence by asking what happens to our subset, partition, or broken permutation of a set  $S$  of numbers if we remove the largest element of  $S$ . Thus it is natural to look for a recurrence to count the number of partitions of  $k$  into  $n$  parts by doing something similar. Unfortunately, since we are counting distributions in which all the objects are identical, there is no way for us to identify a largest element. However if we think geometrically, we can ask what we could remove from a Young diagram to get a Young diagram. Two natural ways to get a partition of a smaller integer from a partition of  $n$  would be to remove the top row of the Young diagram of the partition and to remove the left column of the Young diagram of the partition. These two operations correspond to removing the largest part from the partition and to subtracting 1 from each part of the partition respectively. Even though they are symmetric with respect to conjugation, they aren't symmetric with respect to the number of parts. Thus one might be much more useful than

the other for finding a recurrence for the number of partitions of  $k$  into  $n$  parts.

- 169. In this problem we will study the two operations and see which one seems more useful for getting a recurrence for  $P(k, n)$ .
- (a) How many parts does the remaining partition have when we remove the largest part (more precisely, we reduce its multiplicity by one) from a partition of  $k$  into  $n$  parts? What can you say about the number of parts of the remaining partition if we remove one from each part?
  - (b) If the largest part of a partition is  $j$  and we remove it, what integer is being partitioned by the remaining parts of the partition? If we remove one from each part of a partition of  $k$  into  $n$  parts, what integer is being partitioned by the remaining parts?
  - (c) The last two questions are designed to get you thinking about how we can get a bijection between the set of partitions of  $k$  into  $n$  parts and some other set of partitions that are partitions of a smaller number. These questions describe two different strategies for getting that set of partitions of a smaller number or of smaller numbers. Each strategy leads to a bijection between partitions of  $k$  into  $n$  parts and a set of partitions of a smaller number or numbers. For each strategy, use the answers to the last two questions to find and describe this set of partitions into a smaller number and a bijection between partitions of  $k$  into  $n$  parts and partitions of the smaller integer or integers into appropriate numbers of parts.
  - (d) Find a recurrence (which need not have just two terms on the right hand side) that describes how to compute  $P(k, n)$  in terms of the number of partitions of smaller integers into a smaller number of parts. (Hint: One of the two sets of partitions of smaller numbers from the previous part is more amenable to finding a recurrence than the other.)
  - (e) What is  $P(k, 1)$  for a positive integer  $k$ ?
  - (f) What is  $P(k, k)$  for a positive integer  $k$ ?
  - (g) Use your recurrence to compute a table with the values of  $P(k, n)$  for values of  $k$  between 1 and 7.
  - (h) What would you want to fill into row 0 and column 0 of your table in order to make it consistent with your recurrence? What

does this say  $P(0,0)$  should be? We usually define a sum with no terms in it to be zero. Is that consistent with the way the recurrence says we should define  $P(0,0)$ ?

It is remarkable that there is no known formula for  $P(k, n)$ , nor is there one for  $P(k)$ . This section was devoted to developing methods for computing values of  $P(n, k)$  and finding properties of  $P(n, k)$  that we can prove even without knowing a formula. Some future sections will attempt to develop other methods.

We have seen that the number of partitions of  $k$  into  $n$  parts is equal to the number of ways to distribute  $k$  identical objects to  $n$  recipients so that each receives at least one. If we relax the condition that each recipient receives at least one, then we see that the number of distributions of  $k$  identical objects to  $n$  recipients is  $\sum_{i=1}^n P(k, i)$  because if some recipients receive nothing, it does not matter which recipients these are. This completes rows 7 and 8 of our table of distribution problems. The completed table is shown in Figure 3.2. Every entry in that table tells us how to count something. There are quite a few theorems that you have proved which are summarized by Table 3.2. It would be worthwhile to try to write them all down! The methods we used to complete Figure 3.2 are extensions of the basic counting principles we learned in Chapter 1. The remaining chapters of this book develop more sophisticated kinds of tools that let us solve more sophisticated kinds of counting problems.

### 3.3.4 Partitions into distinct parts

Often  $Q(k, n)$  is used to denote the number of partitions of  $k$  into distinct parts, that is, parts that are different from each other.

170. Show that

$$Q(k, n) \leq \frac{1}{n!} \binom{k-1}{n-1}.$$

- 171. Show that the number of partitions of seven into three parts equals the number of partitions of 10 into three distinct parts.
- 172. There is a relationship between  $P(k, n)$  and  $Q(m, n)$  for some other number  $m$ . Find the number  $m$  that gives you the nicest possible relationship.

Table 3.2: The number of ways to distribute  $k$  objects to  $n$  recipients, with restrictions on how the objects are received

The Twentyfold Way: A Table of Distribution Problems		
$k$ objects and conditions on how they are received	$n$ recipients and mathematical model for distribution	
	Distinct	Identical
1. Distinct no conditions	$n^k$ functions	$\sum_{i=1}^k S(n, i)$ set partitions ( $\leq n$ parts)
2. Distinct Each gets at most one	$n^{\underline{k}}$ $k$ -element permutations	1 if $k \leq n$ ; 0 otherwise
3. Distinct Each gets at least one	$S(k, n)n!$ onto functions	$S(k, n)$ set partitions ( $n$ parts)
4. Distinct Each gets exactly one	$k! = n!$ permutations	1 if $k = n$ ; 0 otherwise
5. Distinct, order matters	$(k + n - 1)^{\underline{k}}$ ordered functions	$\sum_{i=1}^n L(k, i)$ broken permutations ( $\leq n$ parts)
6. Distinct, order matters Each gets at least one	$(k)^{\underline{n}}(k - 1)^{\underline{k-n}}$ ordered onto functions	$L(k, n) = \binom{k}{n}(k - 1)^{\underline{k-n}}$ broken permutations ( $n$ parts)
7. Identical no conditions	$\binom{n+k-1}{k}$ multisets	$\sum_{i=1}^n P(k, i)$ number partitions ( $\leq n$ parts)
8. Identical Each gets at most one	$\binom{n}{k}$ subsets	1 if $k \leq n$ ; 0 otherwise
9. Identical Each gets at least one	$\binom{k-1}{n-1}$ compositions ( $n$ parts)	$P(k, n)$ number partitions ( $n$ parts)
10. Identical Each gets exactly one	1 if $k = n$ ; 0 otherwise	1 if $k = n$ ; 0 otherwise

- 173. Find a recurrence that expresses  $Q(k, n)$  as a sum of  $Q(k - n, m)$  for appropriate values of  $m$ .
- \*174. Show that the number of partitions of  $k$  into distinct parts equals the number of partitions of  $k$  into odd parts.
- \*175. Euler showed that if  $k \neq \frac{3j^2 + j}{2}$ , then the number of partitions of  $k$  into an even number of distinct parts is the same as the number of partitions of  $k$  into an odd number of distinct parts. Prove this, and in the exceptional case find out how the two numbers relate to each other.

### 3.3.5 Supplementary Problems

1. Answer each of the following questions with  $n^k$ ,  $k^n$ ,  $n!$ ,  $k!$ ,  $\binom{n}{k}$ ,  $\binom{k}{n}$ ,  $n^{\overline{k}}$ ,  $k^{\overline{n}}$ ,  $n^{\overline{k}}$ ,  $k^{\overline{n}}$ ,  $\binom{n+k-1}{k}$ ,  $\binom{n+k-1}{n}$ ,  $\binom{n-1}{k-1}$ ,  $\binom{k-1}{n-1}$ , or “none of the above.”
  - (a) In how many ways may we pass out  $k$  identical pieces of candy to  $n$  children?
  - (b) In how many ways may we pass out  $k$  distinct pieces of candy to  $n$  children?
  - (c) In how many ways may we pass out  $k$  identical pieces of candy to  $n$  children so that each gets at most one? (Assume  $k \leq n$ .)
  - (d) In how many ways may we pass out  $k$  distinct pieces of candy to  $n$  children so that each gets at most one? (Assume  $k \leq n$ .)
  - (e) In how many ways may we pass out  $k$  distinct pieces of candy to  $n$  children so that each gets at least one? (Assume  $k \geq n$ .)
  - (f) In how many ways may we pass out  $k$  identical pieces of candy to  $n$  children so that each gets at least one? (Assume  $k \geq n$ .)
  
2. The neighborhood betterment committee has been given  $r$  trees to distribute to  $s$  families living along one side of a street. Unless otherwise specified, it doesn't matter where a family plants the trees it gets.
  - (a) In how many ways can they distribute all of them if the trees are distinct, there are more families than trees, and each family can get at most one?
  - (b) In how many ways can they distribute all of them if the trees are distinct and any family can get any number?
  - (c) In how many ways can they distribute all the trees if the trees are identical, there are no more trees than families, and any family receives at most one?
  - (d) In how many ways can they distribute them if the trees are distinct, there are more trees than families, and each family receives at most one (so there could be some leftover trees)?
  - (e) In how many ways can they distribute all the trees if they are identical and anyone may receive any number of trees?
  - (f) In how many ways can all the trees be distributed and planted if the trees are distinct, any family can get any number, and a family must plant its trees in an evenly spaced row along the road?

- (g) Answer the question in Part 2f assuming that every family must get a tree.
- (h) Answer the question in Part 2e assuming that each family must get at least one tree.
3. In how many ways can  $n$  identical chemistry books,  $r$  identical mathematics books,  $s$  identical physics books, and  $t$  identical astronomy books be arranged on three bookshelves? (Assume there is no limit on the number of books per shelf.)
- 4. One formula for the Lah numbers is

$$L(k, n) = \binom{k}{n} (k-1)^{\overline{k-n}}$$

Find a proof that explains this product.

5. What is the number of partitions of  $n$  into two parts?
6. What is the number of partitions of  $k$  into  $k-2$  parts?
7. Show that the number of partitions of  $k$  into  $n$  parts of size at most  $m$  equals the number of partitions of  $mn-k$  into no more than  $n$  parts of size at most  $m-1$ .
8. Show that the number of partitions of  $k$  into parts of size at most  $m$  is equal to the number of partitions of  $k+m$  into  $m$  parts.
9. You can say something pretty specific about self-conjugate partitions of  $k$  into distinct parts. Figure out what it is and prove it. With that, you should be able to find a relationship between these partitions and partitions whose parts are consecutive integers, starting with 1. What is that relationship?
10. What is  $s(k, 1)$ ?
11. Show that the Stirling numbers of the second kind satisfy the recurrence

$$S(k, n) = \sum_{i=1}^k S(k-i, n-1) \binom{k-1}{i-1}.$$



- 12. Let  $c(k, n)$  be the number of ways for  $k$  children to hold hands to form  $n$  circles, where one child clasping his or her hands together and holding them out to form a circle is considered a circle. (Having Mary hold Sam's right hand is different from having Mary hold Sam's left hand.) Find a recurrence for  $c(k, n)$ . Is the family of numbers  $c(k, n)$  related to any of the other families of numbers we have studied? If so, how?
- 13. How many labeled trees on  $n$  vertices have exactly four vertices of degree 1?
- 14. The *degree sequence* of a graph is a list of the degrees of the vertices in nonincreasing order. For example the degree sequence of the first graph in Figure 2.4 is  $(4, 3, 2, 2, 1)$ . For a graph with vertices labeled 1 through  $n$ , the *ordered degree sequence* of the graph is the sequence  $d_1, d_2, \dots, d_n$  in which  $d_i$  is the degree of vertex  $i$ . For example the ordered degree sequence of the first graph in Figure 2.2 is  $(1, 2, 3, 3, 1, 1, 2, 1)$ .
- (a) How many labeled trees are there on  $n$  vertices with ordered degree sequence  $d_1, d_2, \dots, d_n$ ?
- \* (b) How many labeled trees are there on  $n$  vertices with with the degree sequence in which the degree  $d$  appears  $i_d$  times?



## Chapter 4

# Generating Functions

### 4.1 The Idea of Generating Functions

#### 4.1.1 Visualizing Counting with Pictures

Suppose you are going to choose three pieces of fruit from among apples, pears and bananas for a snack. We can symbolically represent all your choices as

$$\heartsuit\heartsuit\heartsuit + \heartsuit\heartsuit\triangle + \heartsuit\heartsuit\clubsuit + \heartsuit\heartsuit\triangle + \heartsuit\heartsuit\clubsuit + \heartsuit\triangle\triangle + \heartsuit\triangle\clubsuit + \heartsuit\clubsuit\clubsuit + \triangle\clubsuit\clubsuit + \triangle\clubsuit\clubsuit.$$

Here we are using a picture of a piece of fruit to stand for taking a piece of that fruit. Thus  $\heartsuit$  stands for taking an apple,  $\heartsuit\triangle$  for taking an apple and a pear, and  $\heartsuit\heartsuit$  for taking two apples. You can think of the plus sign as standing for the “exclusive or,” that is,  $\heartsuit + \clubsuit$  would stand for “I take an apple or a banana but not both.” To say “I take both an apple and a banana,” we would write  $\heartsuit\clubsuit$ . We can extend the analogy to mathematical notation by condensing our statement that we take three pieces of fruit to

$$\heartsuit^3 + \triangle^3 + \clubsuit^3 + \heartsuit^2\triangle + \heartsuit^2\clubsuit + \heartsuit\triangle^2 + \triangle^2\clubsuit + \heartsuit\clubsuit^2 + \triangle\clubsuit^2 + \heartsuit\triangle\clubsuit.$$

In this notation  $\heartsuit^3$  stands for taking a multiset of three apples, while  $\heartsuit^2\clubsuit$  stands for taking a multiset of two apples and a banana, and so on. What our notation is really doing is giving us a convenient way to list all three element multisets chosen from the set  $\{\heartsuit, \triangle, \clubsuit\}$ .<sup>1</sup>

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<sup>1</sup>This approach was inspired by George Pólya’s paper “Picture Writing,” in the December, 1956 issue of the *American Mathematical Monthly*, page 689. While we are taking a somewhat more formal approach than Pólya, it is still completely in the spirit of his work.

Suppose now that we plan to choose between one and three apples, between one and two pears, and between one and two bananas. In a somewhat clumsy way we could describe our fruit selections as

$$\begin{aligned} \heartsuit\triangle\clubsuit + \heartsuit^2\triangle\clubsuit + \dots + \heartsuit^2\triangle^2\clubsuit + \dots + \heartsuit^2\triangle^2\clubsuit^2 + \heartsuit^3\triangle\clubsuit + \dots + \heartsuit^3\triangle^2\clubsuit + \dots + \heartsuit^3\triangle^2\clubsuit^2. \end{aligned} \quad (4.1)$$

- 176. Using an  $A$  in place of the picture of an apple, a  $P$  in place of the picture of a pear, and a  $B$  in place of the picture of a banana, write out the formula similar to Formula 4.1 without any dots for left out terms. (You may use pictures instead of letters if you prefer, but it gets tedious quite quickly!) Now expand the product  $(A+A^2+A^3)(P+P^2)(B+B^2)$  and compare the result with your formula.
- 177. Substitute  $x$  for all of  $A$ ,  $P$  and  $B$  (or for the corresponding pictures) in the formula you got in Problem 176 and expand the result in powers of  $x$ . Give an interpretation of the coefficient of  $x^n$ .

If we were to expand the formula

$$(\heartsuit + \heartsuit^2 + \heartsuit^3)(\triangle + \triangle^2)(\clubsuit + \clubsuit^2), \quad (4.2)$$

we would get Formula 4.1. Thus Formula 4.1 and Formula 4.2 each describe the number of multisets we can choose from the set  $\{\heartsuit, \triangle, \clubsuit\}$  in which  $\heartsuit$  appears between one and three times, and  $\triangle$  and  $\clubsuit$  each appear once or twice. We interpret Formula 4.1 as describing each individual multiset we can choose, and we interpret Formula 4.2 as saying that we first decide how many apples to take, and then decide how many pears to take, and then decide how many bananas to take. At this stage it might seem a bit magical that doing ordinary algebra with the second formula yields the first, but in fact we could define addition and multiplication with these pictures more formally so we could explain in detail why things work out. However, since the pictures are for motivation, and are actually difficult to write out on paper, it doesn't make much sense to work out these details. We will see an explanation in another context later on.

### 4.1.2 Picture functions

As you've seen, in our descriptions of ways of choosing fruits, we've treated the pictures of the fruit as if they are variables. You've also likely noticed that it is much easier to do algebraic manipulations with letters rather than pictures, simply because it is time consuming to draw the same picture over

and over again, while we are used to writing letters quickly. In the theory of generating functions, we associate variables or polynomials or even power series with members of a set. There is no standard language describing how we associate variables with members of a set, so we shall invent some. By a *picture* of a member of a set we will mean a variable, or perhaps a product of powers of variables (or even a sum of products of powers of variables). A function that assigns a picture  $P(s)$  to each member  $s$  of a set  $S$  will be called a *picture function*. The **picture enumerator** for a picture function  $P$  defined on a set  $S$  will be the sum of the pictures of the elements in  $S$ . In symbols we can write this conveniently as.

$$E_P(S) = \sum_{s:s \in S} P(s).$$

We choose this language because the picture enumerator lists, or enumerates, all the elements of  $S$  according to their pictures. Thus Formula 4.1 is the picture enumerator of the set of all multisets of fruit with between one and three apples, one and two pears, and one and two bananas.

- 178. How would you write down a polynomial in the variable  $A$  that says you should take between zero and three apples?
- 179. How would you write down a picture enumerator that says we take between zero and three apples, between zero and three pears, and between zero and three bananas?
- 180. (Used in Chapter 6.) Notice that when we used  $A^2$  to stand for taking two apples, and  $P^3$  to stand for taking three pears, then we used the product  $A^2P^3$  to stand for taking two apples and three pears. Thus we have chosen the picture of the ordered pair (2 apples, 3 pears) to be the product of the pictures of a multiset of two apples and a multiset of three pears. Show that if  $S_1$  and  $S_2$  are sets with picture functions  $P_1$  and  $P_2$  defined on them, and if we define the picture of an ordered pair  $(x_1, x_2) \in S_1 \times S_2$  to be  $P((x_1, x_2)) = P_1(x_1)P_2(x_2)$ , then the picture enumerator of  $P$  on the set  $S_1 \times S_2$  is  $E_{P_1}(S_1)E_{P_2}(S_2)$ . We call this the **product principle for picture enumerators**.

### 4.1.3 Generating functions

- 181. Suppose you are going to choose a snack of between zero and three apples, between zero and three pears, and between zero and three bananas. Write down a polynomial in one variable  $x$  such that the

coefficient of  $x^n$  is the number of ways to choose a snack with  $n$  pieces of fruit. Hint: substitute something for  $A$ ,  $P$  and  $B$  in your formula from Problem 179.

- 182. Suppose an apple costs 20 cents, a banana costs 25 cents, and a pear costs 30 cents. What should you substitute for  $A$ ,  $P$ , and  $B$  in Problem 179 in order to get a polynomial in which the coefficient of  $x^n$  is the number of ways to choose a selection of fruit that costs  $n$  cents?
- 183. Suppose an apple has 40 calories, a pear has 60 calories, and a banana has 80 calories. What should you substitute for  $A$ ,  $P$ , and  $B$  in Problem 179 in order to get a polynomial in which the coefficient of  $x^n$  is the number of ways to choose a selection of fruit with a total of  $n$  calories?
- 184. We are going to choose a subset of the set  $[n] = \{1, 2, \dots, n\}$ . Suppose we use  $x_1$  to be the picture of choosing 1 to be in our subset. What is the picture enumerator for either choosing 1 or not choosing 1? Suppose that for each  $i$  between 1 and  $n$ , we use  $x_i$  to be the picture of choosing  $i$  to be in our subset. What is the picture enumerator for either choosing  $i$  or not choosing  $i$  to be in our subset? What is the picture enumerator for all possible choices of subsets of  $[n]$ ? What should we substitute for  $x_i$  in order to get a polynomial in  $x$  such that the coefficient of  $x^k$  is the number of ways to choose a  $k$ -element subset of  $n$ ? What theorem have we just reproved (a special case of)?

In Problem 184 we see that we can think of the process of expanding the polynomial  $(1+x)^n$  as a way of “generating” the binomial coefficients  $\binom{n}{k}$  as the coefficients of  $x^k$  in the expansion of  $(1+x)^n$ . For this reason, we say that  $(1+x)^n$  is the “generating function” for the binomial coefficients  $\binom{n}{k}$ . More generally, the **generating function** for a sequence  $a_i$ , defined for  $i$  with  $0 \leq i \leq n$  is the expression  $\sum_{i=0}^n a_i x^i$ , and the **generating function** for the sequence  $a_i$  with  $i \geq 0$  is the expression  $\sum_{i=0}^{\infty} a_i x^i$ . This last expression is an example of a power series. In calculus it is important to think about whether a power series converges in order to determine whether or not it represents a function. In a nice twist of language, even though we use the phrase generating function as the name of a power series in combinatorics, we don’t require the power series to actually represent a function in the usual sense, and so we don’t have to worry about convergence.<sup>2</sup> Instead we think of a power series as a convenient way of representing the terms of a sequence

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<sup>2</sup>In the evolution of our current mathematical terminology, the word function evolved

of numbers of interest to us. The only justification for saying that such a representation is convenient is because of the way algebraic properties of power series capture some of the important properties of some sequences that are of combinatorial importance. The remainder of this chapter is devoted to giving examples of how the algebra of power series reflects combinatorial ideas.

Because we choose to think of power series as strings of symbols that we manipulate by using the ordinary rules of algebra and we choose to ignore issues of convergence, we have to avoid manipulating power series in a way that would require us to add infinitely many real numbers. For example, we cannot make the substitution of  $y + 1$  for  $x$  in the power series  $\sum_{i=0}^{\infty} x^i$ , because in order to interpret  $\sum_{i=0}^{\infty} (y + 1)^i$  as a power series we would have to apply the binomial theorem to each of the  $(y + 1)^i$  terms, and then collect like terms, giving us infinitely many ones added together as the coefficient of  $y^0$ , and in fact infinitely many numbers added together for the coefficient of any  $y^i$ . (On the other hand, it would be fine to substitute  $y + y^2$  for  $x$ . Can you see why?)

#### 4.1.4 Power series

For now, most of our uses of power series will involve just simple algebra. Since we use power series in a different way in combinatorics than we do in calculus, we should review a bit of the algebra of power series.

- 185. In the polynomial  $(a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2 + b_3x^3)$ , what is the coefficient of  $x^2$ ? What is the coefficient of  $x^4$ ?
- 186. In Problem 185 why is there a  $b_0$  and a  $b_1$  in your expression for the coefficient of  $x^2$  but there is not a  $b_0$  or a  $b_1$  in your expression for the coefficient of  $x^4$ ? What is the coefficient of  $x^4$  in

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4)(b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4)?$$

Express this coefficient in the form

$$\sum_{i=0}^4 \text{something},$$

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through several meanings, starting with very imprecise meanings and ending with our current rather precise meaning. The terminology “generating function” may be thought of as an example of one of the earlier usages of the term function.

where the something is an expression you need to figure out. Now suppose that  $a_3 = 0$ ,  $a_4 = 0$ , and  $b_4 = 0$ . To what is your expression equal after you substitute these values? In particular, what does this have to do with Problem 185?

- 187. The point of the Problems 185 and 186 is that so long as we are willing to assume  $a_i = 0$  for  $i > n$  and  $b_j = 0$  for  $j > m$ , then there is a very nice formula for the coefficient of  $x^k$  in the product

$$\left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{j=0}^m b_j x^j \right).$$

Write down this formula explicitly.

- 188. Assuming that the rules you use to do arithmetic with polynomials apply to power series, write down a formula for the coefficient of  $x^k$  in the product

$$\left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right).$$

We use the expression you obtained in Problem 188 to *define* the product of power series. That is, we define the product

$$\left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right)$$

to be the power series  $\sum_{k=0}^{\infty} c_k x^k$ , where  $c_k$  is the expression you found in Problem 188. Since you derived this expression by using the usual rules of algebra for polynomials, it should not be surprising that the product of power series satisfies these rules.<sup>3</sup>

#### 4.1.5 Product principle for generating functions

Each time that we converted a picture function to a generating function by substituting  $x$  or some power of  $x$  for each picture, the coefficient of  $x$  had a meaning that was significant to us. For example, with the picture enumerator for selecting between zero and three each of apples, pears, and

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<sup>3</sup>Technically we should explicitly state these rules and prove that they are all valid for power series multiplication, but it seems like overkill at this point to do so!



bananas, when we substituted  $x$  for each of our pictures, the exponent  $i$  in the power  $x^i$  is the number of pieces of fruit in the fruit selection that led us to  $x^i$ . After we simplify our product by collecting together all like powers of  $x$ , the coefficient of  $x^i$  is the number of fruit selections that use  $i$  pieces of fruit. In the same way, if we substitute  $x^c$  for a picture, where  $c$  is the number of calories in that particular kind of fruit, then the  $i$  in an  $x^i$  term in our generating function stands for the number of calories in a fruit selection that gave rise to  $x^i$ , and the coefficient of  $x^i$  in our generating function is the number of fruit selections with  $i$  calories. The product principle of picture enumerators translates directly into a product principle for generating functions. However, it is possible to give a proof that does not rely on the product principle for enumerators.

- 189. Suppose that we have two sets  $S_1$  and  $S_2$ . Let  $v_1$  ( $v$  stands for value) be a function from  $S_1$  to the nonnegative integers and let  $v_2$  be a function from  $S_2$  to the nonnegative integers. Define a new function  $v$  on the set  $S_1 \times S_2$  by  $v(x_1, x_2) = v_1(x_1) + v_2(x_2)$ . Suppose further that  $\sum_{i=0}^{\infty} a_i x^i$  is the generating function for the number of elements  $x_1$  of  $S_1$  of value  $i$ , that is, with  $v_1(x_1) = i$ . Suppose also that  $\sum_{j=0}^{\infty} b_j x^j$  is the generating function for the number of elements  $x_2$  of  $S_2$  of value  $j$ , that is, with  $v_2(x_2) = j$ . Prove that the coefficient of  $x^k$  in

$$\left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right)$$

is the number of ordered pairs  $(x_1, x_2)$  in  $S_1 \times S_2$  with total value  $k$ , that is, with  $v_1(x_1) + v_2(x_2) = k$ . This is called the **product principle for generating functions**.

Problem 189 may be extended by mathematical induction to prove our next theorem.

**Theorem 7** *If  $S_1, S_2, \dots, S_n$  are sets with a value function  $v_i$  from  $S_i$  to the nonnegative integers for each  $i$ , and  $f_i(x)$  is the generating function for the number of elements of  $S_i$  of each possible value, then the generating function for the number of  $n$ -tuples of each possible total value is  $\prod_{i=1}^n f_i(x)$ .*

- 190. Let  $i$  denote an integer between 1 and  $n$ .
  - (a) What is the generating function for the number of subsets of the singleton set  $\{i\}$  of each possible size? (Notice that the only subsets of  $\{i\}$  are  $\emptyset$  and  $\{i\}$ .)

- (b) Use the product principle for generating functions to prove the binomial theorem.

#### 4.1.6 The extended binomial theorem and multisets

- 191. Suppose once again that  $i$  is an integer between 1 and  $n$ .
- (a) What is the generating function in which the coefficient of  $x^k$  is one? This series is an example of what is called an *infinite geometric series*. In the next part of this problem it will be useful to interpret the coefficient one as the number of multisets of size  $k$  chosen from the singleton set  $\{i\}$ . Namely, there is only one way to choose a multiset of size  $k$  from  $\{i\}$ , namely choose  $i$  exactly  $k$  times.
- (b) Express the generating function in which the coefficient of  $x^k$  is the number of  $k$ -element multisets chosen from  $[n]$  as a power of a power series. What does Problem 123 (in which your answer could be expressed as a binomial coefficient) tell you about what this generating function equals?
- 192. What is the product  $(1-x)\sum_{k=0}^n x^k$ ? What is the product

$$(1-x)\sum_{k=0}^{\infty} x^k?$$

- 193. Express the generating function for the number of multisets of size  $k$  chosen from  $[n]$  (where  $n$  is fixed but  $k$  can be any nonnegative integer) as a 1 over something relatively simple.
- 194. Find a formula for  $(1+x)^{-n}$  as a power series whose coefficients involve binomial coefficients. What does this formula tell you about how we should define  $\binom{-n}{k}$  when  $n$  is positive?
- 195. If you define  $\binom{-n}{k}$  in the way you described in Problem 194, you can write down a version of the binomial theorem for  $(x+y)^n$  that is valid for both nonnegative and negative values of  $n$ . Do so. This is called the *extended binomial theorem*. Write down a special case of with  $n$  negative, like  $n = -3$ , to see an interesting surprise that suggests why we do not use this formula later on.

- 196. Write down the generating function for the number of ways to distribute identical pieces of candy to three children so that no child gets more than 4 pieces. Write this generating function as a quotient of polynomials. Using both the extended binomial theorem and the original binomial theorem, find out in how many ways we can pass out exactly ten pieces.
- 197. What is the generating function for the number of multisets chosen from an  $n$ -element set so that each element appears at least  $j$  times and less than  $m$  times? Write this generating function as a quotient of polynomials, then as a product of a polynomial and a power series.

## 4.2 Generating Functions for Integer Partitions

- 198. If we have five identical pennies, five identical nickels, five identical dimes, and five identical quarters, give the picture enumerator for the combinations of coins we can form and convert it to a generating function for the number of ways to make  $k$  cents with the coins we have. Do the same thing assuming we have an unlimited supply of pennies, nickels, dimes, and quarters.
- 199. Recall that a partition of an integer  $k$  is a multiset of numbers that adds to  $k$ . In Problem 198 we found the generating function for the number of partitions of an integer into parts of size 1, 5, 10, and 25. When working with generating functions for partitions, it is becoming standard to use  $q$  rather than  $x$  as the variable in the generating function. Write your answers to problems involving generating functions for partitions of an integer in this notation.<sup>4</sup>
- Give the generating function for the number partitions of an integer into parts of size one through ten.
  - Give the generating function for the number of partitions of an integer  $k$  into parts of size at most  $m$ , where  $m$  is fixed but  $k$  may vary. Notice this is the generating function for partitions whose Young diagram fits into the space between the line  $x = 0$  and the line  $x = m$  in a coordinate plane. (We assume the boxes in the Young diagram are one unit by one unit.)

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<sup>4</sup>The reason for this change in the notation relates to the subject of finite fields in abstract algebra, where  $q$  is the standard notation for the size of a finite field. While we will make no use of this connection, it will be easier for you to read more advanced work if you get used to the different notation.

- 200. In Problem 199b you gave the generating function for the number of partitions of an integer into parts of size at most  $m$ . Explain why this is also the generating function for partitions of an integer into at most  $m$  parts. Notice that this is the generating function for the number of partitions whose Young diagram fits into the space between the line  $y = 0$  and the line  $y = m$ .
- 201. When studying partitions of integers, it is inconvenient to restrict ourselves to partitions with at most  $m$  parts or partitions with maximum part size  $m$ .
  - (a) Give the generating function for the number of partitions of an integer into parts of any size. Don't forget to use  $q$  rather than  $x$  as your variable.
  - (b) Find the coefficient of  $q^4$  in this generating function.
  - (c) Find the coefficient of  $q^5$  in this generating function.
  - (d) This generating function involves an infinite product. Describe the process you would use to expand this product into as many terms of a power series as you choose.
  - (e) Rewrite any power series that appear in your product as quotients of polynomials or as integers divided by polynomials.
- 202. In Problem 201b, we multiplied together infinitely many power series. Here are two notations for infinite products that look rather similar:

$$\prod_{i=1}^{\infty} 1 + q + q^2 + \cdots + q^i \quad \text{and} \quad \prod_{i=1}^{\infty} 1 + q^i + q^{2i} + \cdots + q^{i^2}.$$

However, one makes sense and one doesn't. Figure out which one makes sense and explain why it makes sense and the other one doesn't. If we want to make sense of a product of the form

$$\prod_{i=1}^{\infty} 1 + p_i(q),$$

where each  $p_i(q)$  is a nonzero polynomial in  $q$ , describe a relatively simple assumption about the polynomials  $p_i(q)$  that will make the product make sense. If we assumed the terms  $p_i(q)$  were nonzero power series, is there a relatively simple assumption we could make about them in order to make the product make sense? (Describe such a condition or explain why you think there couldn't be one.)

- 203. What is the generating function (using  $q$  for the variable) for the number of partitions of an integer in which each part is even?
- 204. What is the generating function (using  $q$  as the variable) for the number of partitions of an integer into distinct parts, that is, in which each part is used at most once?
- 205. Use generating functions to explain why the number of partitions of an integer in which each part is used an even number of times equals the generating function for the number of partitions of an integer in which each part is even. How does this compare to Problem 164?

→•206. Use the fact that

$$\frac{1 - q^{2i}}{1 - q^i} = 1 + q^i$$

and the generating function for the number of partitions of an integer into distinct parts to show how the number of partitions of an integer  $k$  into distinct parts is related to the number of partitions of an integer  $k$  into odd parts.

207. Write down the generating function for the number of ways to partition an integer into parts of size no more than  $m$ , each used an odd number of times. Write down the generating function for the number of partitions of an integer into parts of size no more than  $m$ , each used an even number of times. Use these two generating functions to get a relationship between the two sequences for which you wrote down the generating functions.
- 208. In Problem 199b and Problem 200 you gave the generating functions for, respectively, the number of partitions of  $k$  into parts the largest of which is at most  $m$  and for the number of partitions of  $k$  into at most  $m$  parts. In this problem we will give the generating function for the number of partitions of  $k$  into at most  $n$  parts, the largest of which is at most  $m$ . That is, we will analyze  $\sum_{i=0}^{\infty} a_k q^k$  where  $a_k$  is the number of partitions of  $k$  into at most  $n$  parts, the largest of which is at most  $m$ . Geometrically, it is the generating function for partitions whose Young diagram fits into an  $m$  by  $n$  rectangle, as in Problem 166. This generating function has significant analogs to the binomial coefficient  $\binom{m+n}{n}$ , and so it is denoted by  $\left[ \begin{matrix} m+n \\ n \end{matrix} \right]_q$ . It is called a *q-binomial coefficient*.

- (a) Compute  $\begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \begin{bmatrix} 2+2 \\ 2 \end{bmatrix}_q$ .
- (b) Find explicit formulas for  $\begin{bmatrix} n \\ 1 \end{bmatrix}_q$  and  $\begin{bmatrix} n \\ n-1 \end{bmatrix}_q$ .
- (c) How are  $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$  and  $\begin{bmatrix} m+n \\ m \end{bmatrix}_q$  related? Prove it. (Note this is the same as asking how  $\begin{bmatrix} r \\ s \end{bmatrix}_q$  and  $\begin{bmatrix} r \\ r-s \end{bmatrix}_q$  are related.)
- (d) So far the analogy to  $\binom{m+n}{n}$  is rather thin! If we had a recurrence like the Pascal recurrence, that would demonstrate a real analogy. Is  $\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \begin{bmatrix} m+n-1 \\ n-1 \end{bmatrix}_q + \begin{bmatrix} m+n-1 \\ n \end{bmatrix}_q$ ?
- (e) Recall the two operations we studied in Problem 169.
- i. The largest part of a partition counted by  $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$  is either  $m$  or is less than or equal to  $m-1$ . In the second case, the partition fits into a rectangle that is at most  $m-1$  units wide and at most  $n$  units deep. What is the generating function for partitions of this type? In the first case, what kind of rectangle does the partition we get by removing the largest part sit in? What is the generating function for partitions that sit in this kind of rectangle? What is the generating function for partitions that sit in this kind of rectangle after we remove a largest part of size  $m$ ? What recurrence relation does this give you?
  - ii. What recurrence do you get from the other operation we studied in Problem 169?
  - iii. It is quite likely that the two recurrences you got are different. One would expect that they might give different values for  $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$ . Can you resolve this potential conflict?
- (f) Define  $[n]_q$  to be  $1+q+\cdots+q^{n-1}$  for  $n > 0$  and  $[0]_q = 1$ . We read this simply as  $n$ -sub- $q$ . Define  $[n]!_q$  to be  $[n]_q[n-1]_q \cdots [3]_q[2]_q[1]_q$ . We read this as  $n$  cue-torial, and refer to it as a  $q$ -ary factorial. Show that
- $$\begin{bmatrix} m+n \\ n \end{bmatrix}_q = \frac{[m+n]!_q}{[m]!_q[n]!_q}.$$
- (g) Now think of  $q$  as a variable that we will let approach 1. Find an explicit formula for
- i.  $\lim_{q \rightarrow 1} [n]_q$ .

ii.  $\lim_{q \rightarrow 1} [n]!_q.$

iii.  $\lim_{q \rightarrow 1} \begin{bmatrix} m+n \\ n \end{bmatrix}_q.$

Why is the limit in Part iii equal to the number of partitions (of any number) with at most  $n$  parts all of size most  $m$ ? Can you explain bijectively why this quantity equals the formula you got?

\* (h) What happens to  $\begin{bmatrix} m+n \\ n \end{bmatrix}_q$  if we let  $q$  approach -1?

### 4.3 Generating Functions and Recurrence Relations

Recall that a recurrence relation for a sequence  $a_n$  expresses  $a_n$  in terms of values  $a_i$  for  $i < n$ . For example, the equation  $a_i = 3a_{i-1} + 2^i$  is a first order linear constant coefficient recurrence.

#### 4.3.1 How generating functions are relevant

Algebraic manipulations with generating functions can sometimes reveal the solutions to a recurrence relation.

209. Suppose that  $a_i = 3a_{i-1} + 3^i$ .

- (a) Multiply both sides by  $x^i$  and sum both the left hand side and right hand side from  $i = 1$  to infinity. In the left-hand side use the fact that

$$\sum_{i=1}^{\infty} a_i x^i = \left( \sum_{i=0}^{\infty} a_i x^i \right) - a_0$$

and in the right hand side, use the fact that

$$\sum_{i=1}^{\infty} a_{i-1} x^i = x \sum_{i=1}^{\infty} a_i x^{i-1} = x \sum_{j=0}^{\infty} a_j x^j = x \sum_{i=0}^{\infty} a_i x^i$$

(where we substituted  $j$  for  $i - 1$  to see explicitly how to change the limits of summation, a surprisingly useful trick) to rewrite the equation in terms of the power series  $\sum_{i=0}^{\infty} a_i x^i$ . Solve the resulting equation for the power series  $\sum_{i=0}^{\infty} a_i x^i$ . You can save a lot of writing by using a variable like  $y$  to stand for the power series.

- (b) Use the previous part to get a formula for  $a_i$  in terms of  $a_0$ .
  - (c) Now suppose that  $a_i = 3a_{i-1} + 2^i$ . Repeat the previous two steps for this recurrence relation. (There is a way to do this part using what you already know. Later on we shall introduce yet another way to deal with the kind of generating function that arises here.)
- 210. Suppose we deposit \$5000 in a savings certificate that pays ten percent interest and also participate in a program to add \$1000 to the certificate at the end of each year (from the end of the first year on) that follows (also subject to interest). Assuming we make the \$5000 deposit at the end of year 0, and letting  $a_i$  be the amount of money in the account at the end of year  $i$ , write a recurrence for the amount of money the certificate is worth at the end of year  $n$ . Solve this recurrence. How much money do we have in the account (after our year-end deposit) at the end of ten years? At the end of 20 years?

### 4.3.2 Fibonacci Numbers

The sequence of problems that follows describes a number of hypotheses we might make about a fictional population of rabbits. We use the example of a rabbit population for historic reasons; our goal is a classical sequence of numbers called Fibonacci numbers. When Fibonacci<sup>5</sup> introduced them, he did so with a fictional population of rabbits.

### 4.3.3 Second order linear recurrence relations

- 211. Suppose we start (at the end of month 0) with 10 pairs of baby rabbits, and that after baby rabbits mature for one month they begin to reproduce, with each pair producing two new pairs at the end of each month afterwards. Suppose further that over the time we observe the rabbits, none die. Let  $a_n$  be the number of rabbits we have at the end of month  $n$ . Show that  $a_n = a_{n-1} + 2a_{n-2}$ . This is an example of a *second order linear* recurrence with constant coefficients. Using a method similar to that of Problem 209, show that

$$\sum_{i=0}^{\infty} a_i x^i = \frac{10}{1 - x - 2x^2}.$$

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<sup>5</sup>Apparently Leonardo de Pisa was given the name Fibonacci posthumously. It is a shortening of “son of Bonacci” in Italian.



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This gives us the generating function for the sequence  $a_i$  giving the population in month  $i$ ; shortly we shall see a method for converting this to a solution to the recurrence.

- 212. In Fibonacci's original problem, each pair of mature rabbits produces one new pair at the end of each month, but otherwise the situation is the same as in Problem 211. Assuming that we start with one pair of baby rabbits (at the end of month 0), find the generating function for the number of pairs of rabbits we have at the end on  $n$  months.
- 213. Find the generating function for the solutions to the recurrence

$$a_i = 5a_{i-1} - 6a_{i-2} + 2^i.$$

The recurrence relations we have seen in this section are called *second order* because they specify  $a_i$  in terms of  $a_{i-1}$  and  $a_{i-2}$ , they are called *linear* because  $a_{i-1}$  and  $a_{i-2}$  each appear to the first power, and they are called *constant coefficient recurrences* because the coefficients in front of  $a_{i-1}$  and  $a_{i-2}$  are constants.

#### 4.3.4 Partial fractions

The generating functions you found in the previous section all can be expressed in terms of the reciprocal of a quadratic polynomial. However, without a power series representation, the generating function doesn't tell us what the sequence is. It turns out that whenever you can factor a polynomial into linear factors (and over the complex numbers such a factorization always exists) you can use that factorization to express the reciprocal in terms of power series.

- 214. Express  $\frac{1}{x-3} + \frac{2}{x-2}$  as a single fraction.
- 215. In Problem 214 you see that when we added numerical multiples of the reciprocals of first degree polynomials we got a fraction in which the denominator is a quadratic polynomial. This will always happen unless the two denominators are multiples of each other, because their least common multiple will simply be their product, a quadratic polynomial. This leads us to ask whether a fraction whose denominator is a quadratic polynomial can always be expressed as a sum of fractions

whose denominators are first degree polynomials. Find numbers  $c$  and  $d$  so that

$$\frac{5x + 1}{(x - 3)(x + 5)} = \frac{c}{x - 3} + \frac{d}{x + 5}.$$

- 216. In Problem 215 you may have simply guessed at values of  $c$  and  $d$ , or you may have solved a system of equations in the two unknowns  $c$  and  $d$ . Given constants  $a$ ,  $b$ ,  $r_1$ , and  $r_2$  (with  $r_1 \neq r_2$ ), write down a system of equations we can solve for  $c$  and  $d$  to write

$$\frac{ax + b}{(x - r_1)(x - r_2)} = \frac{c}{x - r_1} + \frac{d}{x - r_2}.$$

Writing down the equations in Problem 216 and solving them is called the *method of partial fractions*. This method will let you find power series expansions for generating functions of the type you found in Problems 211 to 213. However, you have to be able to factor the quadratic polynomials that are in the denominators of your generating functions.

- 217. Use the method of partial fractions to convert the generating function of Problem 211 into the form

$$\frac{c}{x - r_1} + \frac{d}{x - r_2}.$$

Use this to find a formula for  $a_n$ .

- 218. Use the quadratic formula to find the solutions to  $x^2 + x - 1 = 0$ , and use that information to factor  $x^2 + x - 1$ .
- 219. Use the factors you found in Problem 218 to write

$$\frac{1}{x^2 + x - 1}$$

in the form

$$\frac{c}{x - r_1} + \frac{d}{x - r_2}.$$

(Hint: You can save yourself a tremendous amount of frustrating algebra if you arbitrarily choose one of the solutions and call it  $r_1$  and call the other solution  $r_2$  and solve the problem using these algebraic

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symbols in place of the actual roots.<sup>6</sup> Not only will you save yourself some work, but you will get a formula you could use in other problems. When you are done, substitute in the actual values of the solutions and simplify.)

- 220. (a) Use the partial fractions decomposition you found in Problem 218 to write the generating function you found in Problem 212 in the form

$$\sum_{n=0}^{\infty} a_n x^n$$

and use this to give an explicit formula for  $a_n$ . (Hint: once again it will save a lot of tedious algebra if you use the symbols  $r_1$  and  $r_2$  for the solutions as in Problem 219 and substitute the actual values of the solutions once you have a formula for  $a_n$  in terms of  $r_1$  and  $r_2$ .)

$$\begin{aligned} \sum_{n=0}^{\infty} a_n x^n &= \frac{1}{1-x-x^2} = -\frac{1}{x^2+x-1} \\ &= \frac{1}{\sqrt{5}} \cdot \frac{1}{r_1-x} - \frac{1}{\sqrt{5}} \cdot \frac{1}{r_2-x} \\ &= \frac{1}{r_1\sqrt{5}} \cdot \frac{1}{1-x/r_1} - \frac{1}{r_2\sqrt{5}} \cdot \frac{1}{1-x/r_2} \\ &= \frac{1}{r_1\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{x}{r_1}\right)^n - \frac{1}{r_2\sqrt{5}} \sum_{n=0}^{\infty} \left(\frac{x}{r_2}\right)^n \end{aligned}$$

This gives us that

$$\begin{aligned} a_n &= \frac{1}{\sqrt{5} \cdot r_1^{n+1}} + \frac{1}{\sqrt{5} \cdot r_2^{n+1}} \\ &= \frac{2^{n+1}}{\sqrt{5}(-1+\sqrt{5})^{n+1}} + \frac{2^{n+1}}{\sqrt{5}(-1-\sqrt{5})^{n+1}} \\ &= \frac{2^{n+1}(1+\sqrt{5})^{n+1}}{\sqrt{5} \cdot 4^{n+1}} - \frac{2^{n+1}(1-\sqrt{5})^{n+1}}{\sqrt{5} \cdot 4^{n+1}} \\ &= \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

- (b) When we have  $a_0 = 1$  and  $a_1 = 1$ , i.e. when we start with one pair of baby rabbits, the numbers  $a_n$  are called *Fibonacci Numbers*.

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<sup>6</sup>We use the words roots and solutions interchangeably.

Use either the recurrence or your final formula to find  $a_2$  through  $a_8$ . Are you amazed that your general formula produces integers, or for that matter produces rational numbers? Why does the recurrence equation tell you that the Fibonacci numbers are all integers?

- (c) Explain why there is a real number  $b$  such that, for large values of  $n$ , the value of the  $n$ th Fibonacci number is almost exactly (but not quite) some constant times  $b^n$ . (Find  $b$  and the constant.)
- (d) Find an algebraic explanation (not using the recurrence equation) of what happens to make the square roots of five go away in the general formula for the Fibonacci numbers. Explain why there is a real number  $b$  such that, for large values of  $n$ , the value of the  $n$ th Fibonacci number is almost exactly (but not quite) some constant times  $b^n$ . (Find  $b$  and the constant.)
- \*(e) As a challenge (which the author has not yet done), see if you can find a way to show algebraically (not using the recurrence relation, but rather the formula you get by removing the square roots of five) that the formula for the Fibonacci numbers yields integers.

221. Solve the recurrence  $a_n = 4a_{n-1} - 4a_{n-2}$ .

### 4.3.5 Catalan Numbers

- 222. (a) Using either lattice paths or diagonal lattice paths, explain why the Catalan Number  $c_n$  satisfies the recurrence

$$C_n = \sum_{i=1}^n C_{i-1}C_{n-i}.$$

- (b) Show that if we use  $y$  to stand for the power series  $\sum_{i=0}^{\infty} C_n x^n$ , then we can find  $y$  by solving a quadratic equation. (Hint: does the right hand side of the recurrence remind you of some products you have worked with?) Find  $y$ .
- (c) Taylor's theorem from calculus tells us that the extended binomial theorem

$$(1+x)^r = \sum_{i=0}^{\infty} \binom{r}{i} x^i$$

holds for any number real number  $r$ , where  $\binom{r}{i}$  is defined to be

$$\frac{r^{\underline{i}}}{i!} = \frac{r(r-1)\cdots(r-i+1)}{i!}.$$

Use this and your solution for  $y$  (note that of the two possible values for  $y$  that you get from the quadratic formula, only one gives an actual power series) to get a formula for the Catalan numbers.

#### 4.4 Supplementary Problems

1. What is the generating function for the number of ways to pass out  $k$  pieces of candy from an unlimited supply of identical candy to  $n$  children (where  $n$  is fixed) so that each child gets between three and six pieces of candy (inclusive)? Use the fact that

$$(1 + x + x^2 + x^3)(1 - x) = 1 - x^4$$

to find a formula for the number of ways to pass out the candy.

2. (a) In paying off a mortgage loan with initial amount  $A$ , annual interest rate  $p\%$  (on a monthly basis) with a monthly payment of  $m$ , what recurrence describes the amount owed after  $n$  months of payments in terms of the amount owed after  $n - 1$  months? Some technical details: You make the first payment after one month. The amount of interest included in your monthly payment is  $.01p/12$ . This interest rate is applied to the amount you owed immediately after making your last monthly payment.
  - (b) Find a formula for the amount owed after  $n$  months.
  - (c) Find a formula for the number of months needed to bring the amount owed to zero. Another technical point: If you were to make the standard monthly payment  $m$  in the last month, you might actually end up owing a negative amount of money. Therefore it is ok if the result of your formula for the number of months needed gives a non-integer number of months. The bank would just round up to the next integer and adjust your payment so your balance comes out to zero.
  - (d) What should the monthly payment be to pay off the loan over a period of 30 years?

- 3. We have said that for nonnegative  $i$  and positive  $n$  we want to define  $\binom{-n}{i}$  to be  $\binom{n+i-1}{i}$ . If we want the Pascal recurrence to be valid, how should we define  $\binom{-n}{-i}$  when  $n$  and  $i$  are both positive?
- 4. Find a recurrence relation for the number of ways to divide a convex  $n$ -gon into triangles by means of non-intersecting diagonals. How do these numbers relate to the Catalan numbers?
- 5. How does  $\sum_{k=0}^n \binom{n-k}{k}$  relate to the Fibonacci Numbers?
6. Let  $m$  and  $n$  be fixed. Express the generating function for the number of  $k$ -element multisets of an  $n$ -element set such that no element appears more than  $m$  times as a quotient of two polynomials. Use this expression to get a formula for the number of  $k$ -element multisets of an  $n$ -element set such that no element appears more than  $m$  times.
7. One natural but oversimplified model for the growth of a tree is that all new wood grows from the previous year's growth and is proportional to it in amount. To be more precise, assume that the (total) length of the new growth in a given year is the constant  $c$  times the (total) length of new growth in the previous year. Write down a recurrence for the total length  $a_n$  of all the branches of the tree at the end of growing season  $n$ . Find the general solution to your recurrence relation. Assume that we begin with a one meter cutting of new wood (from the previous year) which branches out and grows a total of two meters of new wood in the first year. What will the total length of all the branches of the tree be at the end of  $n$  years?
- 8. (Relevant to Appendix C) We have some chairs which we are going to paint with red, white, blue, green, yellow and purple paint. Suppose that we may paint any number of chairs red or white, that we may paint at most one chair blue, at most three chairs green, only an even number of chairs yellow, and only a multiple of four chairs purple. In how many ways may we paint  $n$  chairs?
9. What is the generating function for the number of partitions of an integer in which each part is used at most  $m$  times? Why is this also the generating function for partitions in which consecutive parts (in a decreasing list representation) differ by at most  $m$  and the smallest part is also at most  $m$ ?

## Chapter 5

# The Principle of Inclusion and Exclusion

### 5.1 The Size of a Union of Sets

One of our very first counting principles was the sum principle which says that the size of a union of disjoint sets is the sum of their sizes. Computing the size of overlapping sets requires, quite naturally, information about how they overlap. Taking such information into account will allow us to develop a powerful extension of the sum principle known as the “principle of inclusion and exclusion.”

#### 5.1.1 Unions of two or three sets

- 223. In a biology lab study of the effects of basic fertilizer ingredients on plants, 16 plants are treated with potash, 16 plants are treated with phosphate, and among these plants, eight are treated with both phosphate and potash. No other treatments are used. How many plants receive at least one treatment? If 32 plants are studied, how many receive no treatment?
- + 224. Give a formula for the size of the union  $A \cup B$  of two sets  $A$  in terms of the sizes  $|A|$  of  $A$ ,  $|B|$  of  $B$ , and  $|A \cap B|$  of  $A \cap B$ . If  $A$  and  $B$  are subsets of some “universal” set  $U$ , express the size of the complement  $U - (A \cup B)$  in terms of the sizes  $|U|$  of  $U$ ,  $|A|$  of  $A$ ,  $|B|$  of  $B$ , and  $|A \cap B|$  of  $A \cap B$ .
- 225. In Problem 223, there were just two fertilizers used to treat the sam-

ple plants. Now suppose there are three fertilizer treatments, and 15 plants are treated with nitrates, 16 with potash, 16 with phosphate, 7 with nitrate and potash, 9 with nitrate and phosphate, 8 with potash and phosphate and 4 with all three. Now how many plants have been treated? If 32 plants were studied, how many received no treatment at all?

- 226. Give a formula for the size of  $A_1 \cup A_2 \cup A_3$  in terms of the sizes of  $A_1$ ,  $A_2$ ,  $A_3$  and the intersections of these sets.

### 5.1.2 Unions of an arbitrary number of sets

- 227. Conjecture a formula for the size of a union of sets

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^n A_i$$

in terms of the sizes of the sets  $A_i$  and their intersections.

The difficulty of generalizing Problem 226 to Problem 227 is not likely to be one of being able to see what the right conjecture is, but of finding a good notation to express your conjecture. In fact, it would be easier for some people to express the conjecture in words than to express it in a notation. We will describe some notation that will make your task easier. It is similar to the notation

$$E_P(S) = \sum_{s:s \in S} P(s).$$

that we used to stand for the sum of the pictures of the elements of a set  $S$  when we introduced picture enumerators.

Let us define

$$\bigcap_{i:i \in I} A_i$$

to mean the intersection over all elements  $i$  in the set  $I$  of  $A_i$ . Thus

$$\bigcap_{i:i \in \{1,3,4,6\}} = A_1 \cap A_3 \cap A_4 \cap A_6. \quad (5.1)$$

This kind of notation, consisting of an operator with a description underneath of the values of a dummy variable of interest to us, can be extended in many ways. For example

$$\begin{aligned} \sum_{I:I \subseteq \{1,2,3,4\}, |I|=2} |\bigcap_{i \in I} A_i| &= |A_1 \cap A_2| + |A_1 \cap A_3| + |A_1 \cap A_4| \\ &+ |A_2 \cap A_3| + |A_2 \cap A_4| + |A_3 \cap A_4|. \end{aligned} \quad (5.2)$$



- 228. Use notation something like that of Equation 5.1 and Equation 5.2 to express the answer to Problem 227. Note there are many different correct ways to do this problem. Try to write down more than one and choose the nicest one you can. Say why you chose it (because your view of what makes a formula nice may be different from somebody else's). The nicest formula won't necessarily involve all the elements of Equations 5.1 and 5.2.
  
- 229. A group of  $n$  students goes to a restaurant carrying backpacks. The manager invites everyone to check their backpack at the check desk and everyone does. While they are eating, a child playing in the check room randomly moves around the claim check stubs on the backpacks. We will try to compute the probability that, at the end of the meal, at least one student receives his or her own backpack. This probability is the fraction of the total number of ways to return the backpacks in which at least one student gets his or her own backpack back.
  - (a) What is the total number of ways to pass back the backpacks?
  - (b) In how many of the distributions of backpacks to students does at least one student get his or her own backpack? (Hint: For each student, how big is the set of backpack distributions in which that student gets the correct backpack? It might be a good idea to first consider cases with  $n = 3, 4,$  and  $5.$ )
  - (c) What is the probability that at least one student gets the correct backpack?
  - (d) What is the probability that no student gets his or her own backpack?
  - (e) As the number of students becomes large, what does the probability that no student gets the correct backpack approach?

Problem 229 is “classically” called the *hatcheck problem*; the name comes from substituting hats for backpacks. It is also sometimes called the *derangement problem*. A *derangement* of an  $n$ -element set is a permutation of that set (thought of as a bijection) that maps no element of the set to itself. One can think of a way of handing back the backpacks as a permutation  $f$  of the students:  $f(i)$  is the owner of the backpack that student  $i$  receives. Then a derangement is a way to pass back the backpacks so that no student gets his or her own.

### 5.1.3 The Principle of Inclusion and Exclusion

The formula you have given in Problem 228 is often called **the principle of inclusion and exclusion** for unions of sets. The reason is the pattern in which the formula first adds (includes) all the sizes of the sets, then subtracts (excludes) all the sizes of the intersections of two sets, then adds (includes) all the sizes of the intersections of three sets, and so on. Notice that we haven't yet proved the principle. There are a variety of proofs. Perhaps one of the most straightforward (though not the most elegant) is an inductive proof that relies on the fact that

$$A_1 \cup A_2 \cup \cdots \cup A_n = (A_1 \cup A_2 \cup \cdots \cup A_{n-1}) \cup A_n$$

and the formula for the size of a union of two sets.

230. Give a proof of your formula for the principle of inclusion and exclusion.
231. A more elegant proof arises if we ask for a picture enumerator for  $A_1 \cup A_2 \cup \cdots \cup A_n$ . So let us assume  $A$  is a set with a picture function  $P$  defined on it and that each set  $A_i$  is a subset of  $A$ .
- By thinking about how we got the formula for the size of a union, write down instead a conjecture for the picture enumerator of a union. You could use a notation like  $E_P(\bigcap_{i:i \in S} A_i)$  for the picture enumerator of the intersection of the sets  $A_i$  for  $i$  in a subset  $S$  of  $[n]$ .
  - If  $x \in \bigcup_{i=1}^n A_i$ , what is the coefficient of  $P(x)$  in (the inclusion-exclusion side of) your formula for  $E_P(\bigcup_{i=1}^n A_i)$ ? (Hints: let  $T$  be the set of all  $i$  such that  $x \in A_i$ . In terms of  $x$ , what is different about the  $i$  in  $T$  and those not in  $T$ ? You may come to a point where the binomial theorem would be helpful.)
  - If  $x \notin \bigcup_{i=1}^n A_i$ , what is the coefficient of  $P(x)$  in (the inclusion-exclusion side of) your formula for  $E_P(\bigcup_{i=1}^n A_i)$ ?
  - How have you proved your conjecture for the picture enumerator of the union of the sets  $A_i$ ?

- (e) How can you get the formula for the principle of inclusion and exclusion from your formula for the picture enumerator of the union?
232. Frequently when we apply the principle of inclusion and exclusion, we will have a situation like that of Problem 229d. That is, we will have a set  $A$  and subsets  $A_1, A_2, \dots, A_n$  and we will want the size or the probability of the set of elements in  $A$  that are *not* in the union. This set is known as the *complement* of the union of the  $A_i$ s in  $A$ , and is denoted by  $A - \bigcup_{i=1}^n A_i$ , or if  $A$  is clear from context, by  $\overline{\bigcup_{i=1}^n A_i}$ . Give the formula for  $\overline{\bigcup_{i=1}^n A_i}$ . The principle of inclusion and exclusion generally refers to both this formula and the one for the union.

We can find a very elegant way of writing the formula in Problem 232 if we let  $\bigcap_{i:i \in \emptyset} A_i = A$ . For this reason, if we have a family of subsets  $A_i$  of a set  $A$ , we define<sup>1</sup>  $\bigcap_{i:i \in \emptyset} A_i = A$ .

## 5.2 Applications of Inclusion and Exclusion

### 5.2.1 Multisets with restricted numbers of elements

233. In how many ways may we distribute  $k$  identical apples to  $n$  children so that no child gets more than four apples? Compare your result with your result in Problem 196.

### 5.2.2 The Ménage Problem

- 234. A group of  $n$  married couples comes to a group discussion session where they all sit around a round table. In how many ways can they sit so that no person is next to his or her spouse? (Note that two people of the same sex can sit next to each other.)
- \*235. A group of  $n$  married couples comes to a group discussion session where they all sit around a round table. In how many ways can they

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<sup>1</sup>For those interested in logic and set theory, given a family of subsets  $A_i$  of a set  $A$ , we define  $\bigcap_{i:i \in S} A_i$  to be the set of all members  $x$  of  $A$  that are in  $A_i$  for all  $i \in S$ . (Note that this allows  $x$  to be in some other  $A_j$ s as well.) Then if  $S = \emptyset$ , our intersection consists of all members  $x$  of  $A$  that satisfy the statement “if  $i \in \emptyset$ , then  $x \in A_i$ .” But since the hypothesis of the ‘if-then’ statement is false, the statement itself is true for all  $x \in A$ . Therefore  $\bigcap_{i:i \in \emptyset} A_i = A$ .

sit so that no person is next to his or her spouse or a person of the same sex? This problem is called the *ménage problem*. (Hint: Reason somewhat as you did in Problem 234, noting that if the set of couples who do sit side-by-side is nonempty, then the sex of the person at each place at the table is determined once we seat one couple in that set.)

### 5.2.3 Counting onto functions

- 236. Given a function  $f$  from the  $k$ -element set  $K$  to the  $n$ -element set  $[n]$ , we say  $f$  is in the set  $A_i$  if  $f(x) \neq i$  for every  $x$  in  $K$ . How many of these sets does an onto function belong to? What is the number of functions from a  $k$ -element set onto an  $n$ -element set?
- 237. Find a formula for the Stirling number (of the second kind)  $S(k, n)$ .
- 238. If we roll a die eight times, we get a sequence of 8 numbers, the number of dots on top on the first roll, the number on the second roll, and so on.
  - (a) What is the number of ways of rolling the die eight times so that each of the numbers one through six appears at least once in our sequence? To get a numerical answer, you will likely need a computer algebra package.
  - (b) What is the probability that we get a sequence in which all six numbers between one and six appear? To get a numerical answer, you will likely need a computer algebra package.
  - (c) How many times do we have to roll the die to have probability at least one half that all six numbers appear in our sequence. To get a numerical answer, you will likely need a computer algebra package.

### 5.2.4 The chromatic polynomial of a graph

We defined a graph to consist of set  $V$  of elements called vertices and a set  $E$  of elements called edges such that each edge joins two vertices. A *coloring* of a graph by the elements of a set  $C$  (of colors) is an assignment of an element of  $C$  to each vertex of the graph; that is, a function from the vertex set  $V$  of the graph to  $C$ . A coloring is called *proper* if for each edge joining

two distinct vertices<sup>2</sup>, the two vertices it joins have different colors. You may have heard of the famous four color theorem of graph theory that says if a graph may be drawn in the plane so that no two edges cross (though they may touch at a vertex), then the graph has a proper coloring with four colors. Here we are interested in a different, though related, problem: namely, in how many ways may we properly color a graph (regardless of whether it can be drawn in the plane or not) using  $k$  or fewer colors? When we studied trees, we restricted ourselves to connected graphs. (Recall that a graph is connected if, for each pair of vertices, there is a walk between them.) Here, disconnected graphs will also be important to us. Given a graph which might or might not be connected, we partition its vertices into blocks called *connected components* as follows. For each vertex  $v$  we put all vertices connected to it by a walk into a block together. Clearly each vertex is in at least one block, because vertex  $v$  is connected to vertex  $v$  by the trivial walk consisting of the single vertex  $v$  and no edges. To have a partition, each vertex must be in one and only one block. To prove that we have defined a partition, suppose that vertex  $v$  is in the blocks  $B_1$  and  $B_2$ . Then  $B_1$  is the set of all vertices connected by walks to some vertex  $v_1$  and  $B_2$  is the set of all vertices connected by walks to some vertex  $v_2$ .

•239. (Relevant in Appendix C as well as this section.) Show that  $B_1 = B_2$ .

Since  $B_1 = B_2$ , these two sets are the same block, and thus all blocks containing  $v$  are identical, so  $v$  is in only one block. Thus we have a partition of the vertex set, and the blocks of the partition are the connected components of the graph. Notice that the connected components depend on the edge set of the graph. If we have a graph on the vertex set  $V$  with edge set  $E$  and another graph on the vertex set  $V$  with edge set  $E'$ , then these two graphs could have different connected components. It is traditional to use the Greek letter  $\gamma$  (gamma)<sup>3</sup> to stand for the number of connected components of a graph; in particular,  $\gamma(V, E)$  stands for the number of connected components of the graph with vertex set  $V$  and edge set  $E$ . We are going to show how the principle of inclusion and exclusion may be used to compute the number of ways to color a graph properly using colors from a set  $C$  of  $c$  colors.

•240. Suppose we have a graph  $G$  with vertex set  $V$  and edge set  $E =$

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<sup>2</sup>If a graph had a loop connecting a vertex to itself, that loop would connect a vertex to a vertex of the same color. It is because of this that we only consider edges with two distinct vertices in our definition.

<sup>3</sup>The Greek letter gamma is pronounced gam-uh, where gam rhymes with ham.

$\{e_1, e_2, \dots, e_{|E|}\}$ . Suppose  $F$  is a subset of  $E$ . Suppose we have a set  $C$  of  $c$  colors with which to color the vertices.

- (a) In terms of  $\gamma(V, F)$ , in how many ways may we color the vertices of  $G$  so that each edge in  $F$  connects two vertices of the same color?
- (b) Given a coloring of  $G$ , for each edge  $e_i$  in  $E$ , let us consider the set  $A_i$  of colorings that the endpoints of  $e$  are colored the same color. In which sets  $A_i$  does a proper coloring lie?
- (c) Find a formula (which may involve summing over all subsets  $F$  of the edge set of the graph and using the number  $\gamma(V, F)$  of connected components of the graph with vertex set  $V$  and edge set  $F$ ) for the number of proper colorings of  $G$  using colors in the set  $C$ .

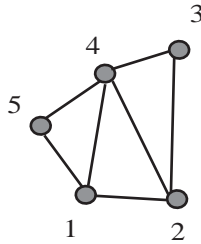
The formula you found in Problem 240c is a formula that involves powers of  $c$ , and so it is a polynomial function of  $c$ . Thus it is called the “chromatic polynomial of the graph  $G$ .” Since we like to think about polynomials as having a variable  $x$  and we like to think of  $c$  as standing for some constant, people often use  $x$  as the notation for the number of colors we are using to color  $G$ . Frequently people will use  $\chi_G(x)$  to stand for the number of ways to color  $G$  with  $x$  colors, and call  $\chi_G(x)$  the *chromatic polynomial* of  $G$ .

### 5.3 Deletion-Contraction and the Chromatic Polynomial

- 241. In Chapter 2 we introduced the deletion-contraction recurrence for counting spanning trees of a graph. Figure out how the chromatic polynomial of a graph is related to those resulting from deletion of an edge  $e$  and from contraction of that same edge  $e$ . Try to find a recurrence like the one for counting spanning trees that expresses the chromatic polynomial of a graph in terms of the chromatic polynomials of  $G - e$  and  $G/e$  for an arbitrary edge  $e$ . Use this recurrence to give another proof that the number of ways to color a graph with  $x$  colors is a polynomial function of  $x$ .
242. Use the deletion-contraction recurrence to compute the chromatic polynomial of the graph in Figure 5.1. (You can simplify your computations by thinking about the effect on the chromatic polynomial of

deleting an edge that is a loop, or deleting one of several edges between the same two vertices.)

Figure 5.1: A graph.



- 243. (a) In how many ways may you properly color the vertices of a path on  $n$  vertices with  $x$  colors? Describe any dependence of the chromatic polynomial of a path on the number of vertices.
- \* (b) (Not tremendously hard.) In how many ways may you properly color the vertices of a cycle on  $n$  vertices with  $x$  colors? Describe any dependence of the chromatic polynomial of a cycle on the number of vertices.
244. In how many ways may you properly color the vertices of a tree on  $n$  vertices with  $x$  colors?
- 245. What do you observe about the signs of the coefficients of the chromatic polynomial of the graph in Figure 5.1? What about the signs of the coefficients of the chromatic polynomial of a path? Of a cycle? Of a tree? Make a conjecture about the signs of the coefficients of a chromatic polynomial and prove it.

## 5.4 Supplementary Problems

1. Each person attending a party has been asked to bring a prize. The person planning the party has arranged to give out exactly as many prizes as there are guests, but any person may win any number of prizes. If there are  $n$  guests, in how many ways may the prizes be given out so that nobody gets the prize that he or she brought?

2. There are  $m$  students attending a seminar in a room with  $n$  seats. The seminar is a long one, and in the middle the group takes a break. In how many ways may the students return to the room and sit down so that nobody is in the same seat as before?
3. What is the number of ways to pass out  $k$  pieces of candy from an unlimited supply of identical candy to  $n$  children (where  $n$  is fixed) so that each child gets between three and six pieces of candy (inclusive)? If you have done Supplementary Problem 1 in Chapter 4 compare your answer in that problem with your answer in this one.
- 4. In how many ways may  $k$  distinct books be arranged on  $n$  shelves so that no shelf gets more than  $m$  books?
- 5. Suppose that  $n$  children join hands in a circle for a game at nursery school. The game involves everyone falling down (and letting go). In how many ways may they join hands in a circle again so that nobody has the same person immediately to the right both times the group joins hands?
- \*6. Suppose that  $n$  people link arms in a folk-dance and dance in a circle. Later on they let go and dance some more, after which they link arms in a circle again. In how many ways can they link arms the second time so that no one links with a person with whom he or she linked arms before?
- \*7. (A challenge; the author has not tried to solve this one!) Redo Problem 6 in the case that there are  $n$  men and  $n$  women and when people arrange themselves in a circle they do so alternating gender.
- 8. Suppose we take two graphs  $G_1$  and  $G_2$  with disjoint vertex sets, we choose one vertex on each graph, and connect these two vertices by an edge  $e$  to get a graph  $G_{12}$ . How does the chromatic polynomial of  $G_{12}$  relate to those of  $G_1$  and  $G_2$ ?



# Chapter 6

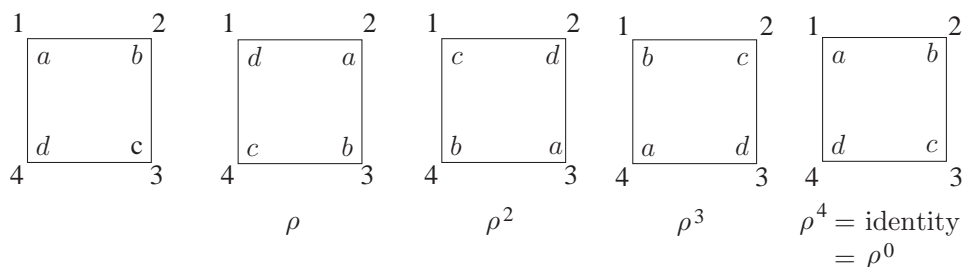
## Groups Acting on Sets

### 6.1 Permutation Groups

Until now we have thought of permutations mostly as ways of listing the elements of a set. In this chapter we will find it very useful to think of permutations as functions. This will help us in using permutations to solve enumeration problems that cannot be solved by the quotient principle because they involve counting the blocks of a partition in which the blocks don't have the same size. We begin by studying the kinds of permutations that arise in situations where we have used the quotient principle in the past.

#### 6.1.1 The rotations of a square

Figure 6.1: The four possible results of rotating a square but maintaining its location.



In Figure 6.1 we show a square with its four vertices labeled  $a$ ,  $b$ ,  $c$ , and  $d$ . We have also labeled the spots in the plane where each of these vertices falls with the label 1, 2, 3, or 4. Then we have shown the effect of rotating the square clockwise through 90, 180, 270, and 360 degrees (which is the same as rotating through 0 degrees). Underneath each of the rotated squares we have named the function that carries out the rotation. We use  $\rho$ , the Greek letter pronounced “row,” to stand for a 90 degree clockwise rotation. We use  $\rho^2$  to stand for two 90 degree rotations, and so on. We can think of the function  $\rho$  as a function on the four element set<sup>1</sup>  $\{1, 2, 3, 4\}$ . In particular, for any function  $\varphi$  (the Greek letter phi, usually pronounced “fee,” but sometimes “fie”) from the plane back to itself that may move the square around but otherwise leaves it in the same location, we let  $\varphi(i)$  be the label of the place where vertex previously in position  $i$  is now. Thus  $\rho(1) = 2$ ,  $\rho(2) = 3$ ,  $\rho(3) = 4$  and  $\rho(4) = 1$ . Notice that  $\rho$  is a permutation on the set  $\{1, 2, 3, 4\}$ .

- 246. The composition  $f \circ g$  of two functions  $f$  and  $g$  is defined by  $f \circ g(x) = f(g(x))$ . Is  $\rho^3$  the composition of  $\rho$  and  $\rho^2$ ? Does the answer depend on the order in which we write  $\rho$  and  $\rho^2$ ? How is  $\rho^2$  related to  $\rho$ ?
- 247. Is the composition of two permutations always a permutation?

In Problem 246 you see that we can think of  $\rho^2 \circ \rho$  as the result of first rotating by 90 degrees and then by another 180 degrees. In other words, the composition of two rotations is the same thing as first doing one and then doing the other. Of course there is nothing special about 90 degrees and 180 degrees. As long as we first do one rotation through a multiple of 90 degrees and then another rotation through a multiple of 90 degrees, the composition of these rotations is a rotation through a multiple of 90 degrees.

If we first rotate by 90 degrees and then by 270 degrees then we have rotated by 360 degrees, which does nothing visible to the square. Thus we say that  $\rho^4$  is the “identity function.” In general the **identity function** on a set  $S$ , denoted by  $\iota$  (the Greek letter iota, pronounced eye-oh-ta) is the function that takes each element of the set to itself. In symbols,  $\iota(x) = x$  for every  $x$  in  $S$ . Of course the identity function on a set is a permutation of that set.

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<sup>1</sup>What we are doing is restricting the rotation  $\rho$  to the set  $\{1, 2, 3, 4\}$ .

### 6.1.2 Groups of permutations

- 248. For any function  $\varphi$  from a set  $S$  to itself, we define  $\varphi^n$  (for nonnegative integers  $n$ ) inductively by  $\varphi^0 = \iota$  and  $\varphi^n = \varphi^{n-1} \circ \varphi$  for every positive integer  $n$ . If  $\varphi$  is a permutation, is  $\varphi^n$  a permutation? Based on your experience with previous inductive proofs, what do you expect  $\varphi^n \circ \varphi^m$  to be? What do you expect  $(\varphi^m)^n$  to be? There is no need to prove these last two answers are correct, for you have, in effect, already done so in Chapter 2.
- 249. If we perform the composition  $\iota \circ \varphi$  for any function  $\varphi$  from  $S$  to  $S$ , what function do we get? What if we perform the composition  $\varphi \circ \iota$ ?

What you have observed about  $\iota$  in Problem 249 is called the *identity property* of  $\iota$ . In the context of permutations, people usually call the function  $\iota$  “the identity” rather than calling it “iota.”

Since rotating first by 90 degrees and then by 270 degrees has the same effect as doing nothing, we can think of the 270 degree rotation as undoing what the 90 degree rotation does. For this reason we say that in the rotations of the square,  $\rho^3$  is the “inverse” of  $\rho$ . In general, a function  $\varphi : T \rightarrow S$  is called an **inverse** of a function  $\sigma : S \rightarrow T$  ( $\sigma$  is the lower case Greek letter sigma) if  $\varphi \circ \sigma = \sigma \circ \varphi = \iota$ . For a slower introduction to inverses and practice with them, see Section A.1.3 in Appendix A. Since a permutation is a bijection, it has a unique inverse, as in Section A.1.3 of Appendix A. And since the inverse of a bijection is a bijection (again, as in the Appendix), the inverse of a permutation is a permutation.

We use  $\varphi^{-1}$  to denote the inverse of the permutation  $\varphi$ . We’ve seen that the rotations of the square are functions that return the square to its original location but may move the vertices to different places. In this way we create permutations of the vertices of the square. We’ve observed three important properties of these permutations.

- (Identity Property) These permutations include the identity permutation.
- (Inverse Property) Whenever these permutations include  $\varphi$ , they also include  $\varphi^{-1}$ .
- (Closure Property) Whenever these permutations include  $\varphi$  and  $\sigma$ , they also include  $\varphi \circ \sigma$ .

A set of permutations with these three properties is called a **permutation group**<sup>2</sup> or a group of permutations. We call the group of permutations corresponding to rotations of the square the *rotation group* of the square. There is a similar rotation group with  $n$  elements for any regular  $n$ -gon.

250. If  $f : S \rightarrow T$ ,  $g : T \rightarrow X$ , and  $h : X \rightarrow Y$ , is

$$h \circ (g \circ f) = (h \circ g) \circ f?$$

What does this say about the status of the *associative law*

$$\rho \circ (\sigma \circ \varphi) = (\rho \circ \sigma) \circ \varphi$$

in a group of permutations?

251. •(a) How should we define  $\varphi^{-n}$  for an element  $\varphi$  of a permutation group?
- (b) Will the two standard rules for exponents

$$a^m a^n = a^{m+n} \text{ and } (a^m)^n = a^{mn}$$

still hold in a group if one or more of the exponents may be negative? (No proof required yet.)

- (c) Proving that  $(\varphi^{-m})^n = \varphi^{-mn}$  when  $m$  and  $n$  are nonnegative is different from proving that  $(\varphi^m)^{-n} = \varphi^{-mn}$  when  $m$  and  $n$  are nonnegative. Make a list of all such formulas we would need to prove in order to prove that the rules of exponents of Part 251b do hold for all nonnegative and negative  $m$  and  $n$ .
- (d) If the rules hold, give enough of the proof to show that you know how to do it; otherwise give a counterexample.
- 252. If a finite set of permutations satisfies the closure property is it a permutation group?
- 253. There are three-dimensional geometric motions of the square that return it to its original location but move some of the vertices to other positions. For example, if we flip the square around a diagonal, most

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<sup>2</sup>The concept of a permutation group is a special case of the concept of a *group* that one studies in abstract algebra. When we refer to a group in what follows, if you know what groups are in the more abstract sense, you may use the word in this way. If you do not know about groups in this more abstract sense, then you may assume we mean permutation group when we say group.

of it moves out of the plane during the flip, but the square ends up in the same location. Draw a figure like Figure 6.1 that shows all the possible results of such motions, including the ones shown in Figure 6.1. Do the corresponding permutations form a group?

→ 254. Let  $\sigma$  and  $\varphi$  be permutations.

- (a) Why must  $\sigma \circ \varphi$  have an inverse?
- (b) Is  $(\sigma \circ \varphi)^{-1} = \sigma^{-1}\varphi^{-1}$ ? (Prove or give a counter-example.)
- (c) Is  $(\sigma \circ \varphi)^{-1} = \varphi^{-1}\sigma^{-1}$ ? (Prove or give a counter-example.)

○ 255. Explain why the set of all permutations of four elements is a permutation group. How many elements does this group have? This group is called the *symmetric group on four letters* and is denoted by  $S_4$ .

### 6.1.3 The symmetric group

In general, the set of all permutations of an  $n$ -element set is a group. It is called the **symmetric group on  $n$  letters**. We don't have nice geometric descriptions (like rotations) for all its elements, and it would be inconvenient to have to write down something like "Let  $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 4,$  and  $\sigma(4) = 1$ " each time we need to introduce a new permutation. We introduce a new notation for permutations that allows us to denote them *reasonably* compactly and compose them reasonably quickly. If  $\sigma$  is the permutation of  $\{1, 2, 3, 4\}$  given by  $\sigma(1) = 3, \sigma(2) = 1, \sigma(3) = 4$  and  $\sigma(4) = 2$ , we write

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}.$$

We call this notation the *two row notation* for permutations. In the two row notation for a permutation of  $\{a_1, a_2, \dots, a_n\}$ , we write the numbers  $a_1$  through  $a_n$  in one row and we write  $\sigma(a_1)$  through  $\sigma(a_n)$  in a row right below, enclosing both rows in parentheses. Notice that

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 1 & 3 & 2 & 4 \end{pmatrix},$$

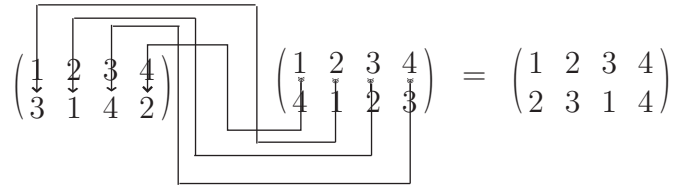
although the second ordering of the columns is rarely used.

If  $\varphi$  is given by

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

then, by applying the definition of composition of functions, we may compute  $\sigma \circ \varphi$  as shown in Figure 6.2.

Figure 6.2: How to multiply permutations in two row notation.



We don't normally put the circle between two permutations in two row notation when we are composing them, and refer to the operation as multiplying the permutations, or as the product of the permutations. To see how Figure 6.2 illustrates composition, notice that the arrow starting at 1 in  $\varphi$  goes to 4. Then from the 4 in  $\varphi$  it goes to the 4 in  $\sigma$  and then to 2. This illustrates that  $\varphi(1) = 4$  and  $\sigma(4) = 2$ , so that  $\sigma(\varphi(1)) = 2$ .

256. For practice, compute  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 5 & 1 & 2 \end{pmatrix}$ .

#### 6.1.4 The dihedral group

We found four permutations that correspond to rotations of the square. In Problem 253 you found four permutations that correspond to flips of the square in space. One flip fixes the vertices in the places labeled 1 and 3 and interchanges the vertices in the places labeled 2 and 4. Let us denote it by  $\varphi_{1|3}$ . One flip fixes the vertices in the positions labeled 2 and 4 and interchanges those in the positions labeled 1 and 3. Let us denote it by  $\varphi_{2|4}$ . One flip interchanges the vertices in the places labeled 1 and 2 and also interchanges those in the places labeled 3 and 4. Let us denote it by  $\varphi_{12|34}$ . The fourth flip interchanges the vertices in the places labeled 1 and 4 and interchanges those in the places labeled 2 and 3. Let us denote it by  $\varphi_{14|23}$ . Notice that  $\varphi_{1|3}$  is a permutation that takes the vertex in place 1 to the vertex in place 1 and the vertex in place 3 to the vertex in place 3, while  $\varphi_{12|34}$  is a permutation that takes the edge between places 1 and 2 to the edge between places 2 and 1 (which is the same edge) and takes the edge between places 3 and 4 to the edge between places 4 and 3 (which is the same edge). This should help to explain the similarity in the notation for the two different kinds of flips.

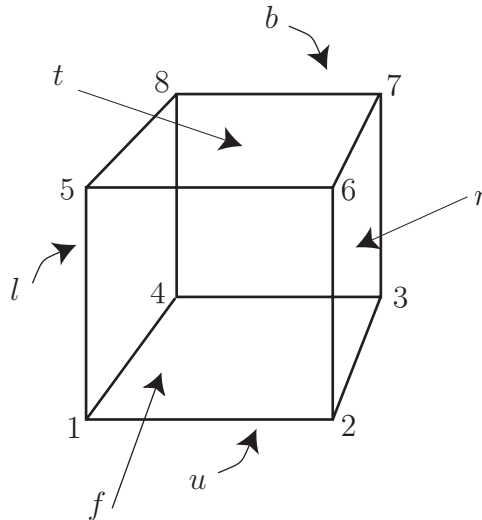
- 257. Write down the two row notation for  $\rho^3$ ,  $\varphi_{2|4}$ ,  $\varphi_{12|34}$  and  $\varphi_{2|4} \circ \varphi_{12|34}$ .

258. (You may have already done this problem in Problem 253, in which case you need not do it again!) In Problem 253, if a rigid motion in three-dimensional space returns the square to its original location, in how many places can vertex number one land? Once the location of vertex number one is decided, how many possible locations are there for vertex two? Once the locations of vertex one and vertex two are decided, how many locations are there for vertex three? Answer the same question for vertex four. What does this say about the relationship between the four rotations and four flips described just before Problem 257 and the permutations you described in Problem 253?

The four rotations and four flips of the square described before Problem 257 form a group called the dihedral group of the square. Sometimes the group is denoted  $D_8$  because it has eight elements, and sometimes the group is denoted by  $D_4$  because it deals with four vertices! Let us agree to use the notation  $D_4$  for the dihedral group of the square. There is a similar **dihedral group**, denoted by  $D_n$ , of all the rigid motions of three-dimensional space that return a regular  $n$ -gon to its original location (but might put the vertices in different places).

259. Another view of the dihedral group of the square is that it is the group of all distance preserving functions, also called *isometries*, from a square to itself. Notice that an isometry must be a bijection. Any rigid motion of the square preserves the distances between all points of the square. However, it is conceivable that there might be some isometries that do not arise from rigid motions. (We will see some later on in the case of a cube.) Show that there are exactly eight isometries (distance preserving functions) from a square to itself.
- 260. How many elements does the group  $D_n$  have? Prove that you are correct.
- 261. In Figure 6.3 we show a cube with the positions of its vertices and faces labeled. As with motions of the square, we let  $\varphi(x)$  be the label of the place where vertex previously in position  $x$  is now.
- (a) Write in two row notation the permutation  $\rho$  of the vertices that corresponds to rotating the cube 90 degrees around a vertical axis through the faces  $t$  (for top) and  $u$  (for underneath). (Rotate in a right-handed fashion around this axis, meaning that vertex 6 goes to the back and vertex 8 comes to the front.)

Figure 6.3: A cube with the positions of its vertices and faces labeled. The curved arrows point to the faces that are blocked by the cube.



- (b) Write in two row notation the permutation  $\varphi$  that rotates the cube 120 degrees around the diagonal from vertex 1 to vertex 7 and carries vertex 8 to vertex 6.
- (c) Compute the two row notation for  $\rho \circ \varphi$ .
- (d) Is the permutation  $\rho \circ \varphi$  a rotation of the cube around some axis? If so, say what the axis is and how many degrees we rotate around the axis. If  $\rho \circ \varphi$  is not a rotation, give a geometric description of it.
- 262. How many permutations are in the group  $R$ ?  $R$  is sometimes called the “rotation group” of the cube. Can you justify this?
- 263. As with a two-dimensional figure, it is possible to talk about isometries of a three-dimensional figure. These are distance preserving functions from the figure to itself. The function that reflects the cube in Figure 6.3 through a plane halfway between the bottom face and top face exchanges the vertices 1 and 5, 2 and 6, 3 and 7, and 4 and 8 of the cube. This function preserves distances between points in the cube. However, it cannot be achieved by a rigid motion of the cube because a rigid motion that takes vertex 1 to vertex 5, vertex 2 to vertex 6,



vertex 3 to vertex 7, and vertex 4 to vertex 8 would not return the cube to its original location; rather it would put the bottom of the cube where its top previously was and would put the rest of the cube above that square rather than below it.

- (a) How many elements are there in the group of permutations of [8] that correspond to isometries of the cube?
- (b) Is every permutation of [8] that corresponds to an isometry either a rotation or a reflection?

### 6.1.5 Group tables (Optional)

We can always figure out the composition of two permutations of the same set by using the definition of composition, but if we are going to work with a given permutation group again and again, it is worth making the computations once and recording them in a table. For example, the group of rotations of the square may be represented as in Table 6.1. We list the elements of our group, with the identity first, across the top of the table and down the left side of the table, using the same order both times. Then in the row labeled by the group element  $\sigma$  and the column labeled by the group element  $\varphi$ , we write the composition  $\sigma \circ \varphi$ , expressed in terms of the elements we have listed on the top and on the left side. Since a group of permutations is closed under composition, the result  $\sigma \circ \varphi$  will always be expressible as one of these elements.

Table 6.1: The group table for the rotations of a square.

$\circ$	$\iota$	$\rho$	$\rho^2$	$\rho^3$
$\iota$	$\iota$	$\rho$	$\rho^2$	$\rho^3$
$\rho$	$\rho$	$\rho^2$	$\rho^3$	$\iota$
$\rho^2$	$\rho^2$	$\rho^3$	$\iota$	$\rho$
$\rho^3$	$\rho^3$	$\iota$	$\rho$	$\rho^2$

264. In Table 6.1, all the entries in a row (not including the first entry, the one to the left of the line) are different. Will this be true in any group table for a permutation group? Why or why not? Also in Table 6.1 all the entries in a column (not including the first entry, the one above the line) are different. Will this be true in any group table for a permutation group? Why or why not?

265. In Table 6.1, every element of the group appears in every row (even if you don't include the first element, the one before the line). Will this be true in any group table for a permutation group? Why or why not? Also in Table 6.1 every element of the group appears in every column (even if you don't include the first entry, the one before the line). Will this be true in any group table for a permutation group? Why or why not?
266. Write down the group table for the dihedral group  $D_4$ . Use the  $\varphi$  notation described earlier to denote the flips. (Hints: Part of the table has already been written down. Will you need to think hard to write down the last row? Will you need to think hard to write down the last column?)

You may notice that the associative law, the identity property, and the inverse property are three of the most important rules that we use in regrouping parentheses in algebraic expressions when solving equations. There is one property we have not yet mentioned, the *commutative law*, which would say that  $\sigma \circ \varphi = \varphi \circ \sigma$ . It is easy to see from the group table of the rotation group of a square that it satisfies the commutative law.

267. Does the commutative law hold in all permutation groups?

### 6.1.6 Subgroups

We have seen that the dihedral group  $D_4$  contains a copy of the group of rotations of the square. When one group  $G$  of permutations of a set  $S$  is a subset of another group  $G'$  of permutations of  $S$ , we say that  $G$  is a **subgroup** of  $G'$ .

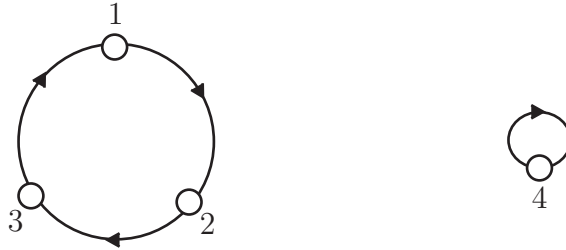
268. Find all subgroups of the group  $D_4$  and explain why your list is complete.
269. Can you find subgroups of the symmetric group  $S_4$  with two elements? Three elements? Four elements? Six elements? (For each positive answer, describe a subgroup. For each negative answer, explain why not.)

### 6.1.7 The cycle decomposition of a permutation

The digraph of a permutation gives us a nice way to think about it. Notice that the product in Figure 6.2 is  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ . We have drawn the

directed graph of this permutation in Figure 6.4. You see that the graph

Figure 6.4: The directed graph of the permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$ .



has two directed cycles, the rather trivial one with vertex 4 pointing to itself, and the nontrivial one with vertex 1 pointing to vertex 2 pointing to vertex 3 which points back to vertex 1. A permutation is called a **cycle** if its digraph consists of exactly one cycle. Thus  $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$  is a cycle but  $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$  is not a cycle by our definition. We write  $(1\ 2\ 3)$  or  $(2\ 3\ 1)$  or  $(3\ 1\ 2)$  to stand for the cycle  $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ .

We can describe cycles in another way as well. A **cycle** of the permutation  $\sigma$  is a list  $(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i))$  that does not have repeated elements while the list  $(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i)\ \sigma^{n+1}(i))$  does have repeated elements.

270. If the list  $(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i))$  does not have repeated elements but the list  $(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i)\ \sigma^{n+1}(i))$  does have repeated elements, then what is  $\sigma^{n+1}(i)$ ?

We say  $\sigma^j(i)$  is an *element* of the cycle  $(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i))$ . Notice that the case  $j = 0$  means  $i$  is an element of the cycle. Notice also that if  $j > n$ ,  $\sigma^j(i) = \sigma^{j-n-1}(i)$ , so the distinct elements of the cycle are  $i, \sigma(i), \sigma^2(i)$ , through  $\sigma^n(i)$ . We think of the cycle  $(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i))$  as representing the permutation  $\sigma$  restricted to the set of elements of the cycle. We say that the cycles

$$(i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^n(i))$$

and

$$(\sigma^j(i)\ \sigma^{j+1}(i)\ \dots\ \sigma^n(i)\ i\ \sigma(i)\ \sigma^2(i)\ \dots\ \sigma^{j-1}(i))$$

are *equivalent*. Equivalent cycles represent the same permutation on the set of elements of the cycle. For this reason, we consider equivalent cycles to be equal in the same way we consider  $\frac{1}{2}$  and  $\frac{2}{4}$  to be equal. In particular, this means that  $(i_1 i_2 \dots i_n) = (i_j i_{j+1} \dots i_n i_1 i_2 \dots i_{j-1})$ .

- 271. Find the cycles of the permutations  $\rho$ ,  $\varphi_{1|3}$  and  $\varphi_{12|34}$  in the group  $D_4$ .
- 272. Find the cycles of the permutation  $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 6 & 2 & 9 & 7 & 1 & 5 & 8 \end{pmatrix}$ .
- 273. If two cycles of  $\sigma$  have an element in common, what can we say about them?

Problem 273 leads almost immediately to the following theorem.

**Theorem 8** *For each permutation  $\sigma$  of a set  $S$ , there is a unique partition of  $S$  each of whose blocks is the set of elements of a cycle of  $\sigma$ .*

More informally, we may say that every permutation partitions its domain into disjoint cycles. We call the set of cycles of a permutation the *cycle decomposition* of the permutation. Since the cycles of a permutation  $\sigma$  tell us  $\sigma(x)$  for every  $x$  in the domain of  $\sigma$ , the cycle decomposition of a permutation completely determines the permutation. Using our informal language, we can express this idea in the following corollary to Theorem 8.

**Corollary 2** *Every partition of a set  $S$  into cycles determines a unique permutation of  $S$ .*

- 274. Prove Theorem 8.

In Problems 271 and 272 you found the cycle decompositions of typical elements of the group  $D_4$  and of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 6 & 2 & 9 & 7 & 1 & 5 & 8 \end{pmatrix}.$$

The group of all rotations of the square is simply the set of the four powers of the cycle  $\rho = (1\ 2\ 3\ 4)$ . For this reason, it is called a *cyclic group*<sup>3</sup> and often denoted by  $C_4$ . Similarly, the rotation group of an  $n$ -gon is usually denoted by  $C_n$ .

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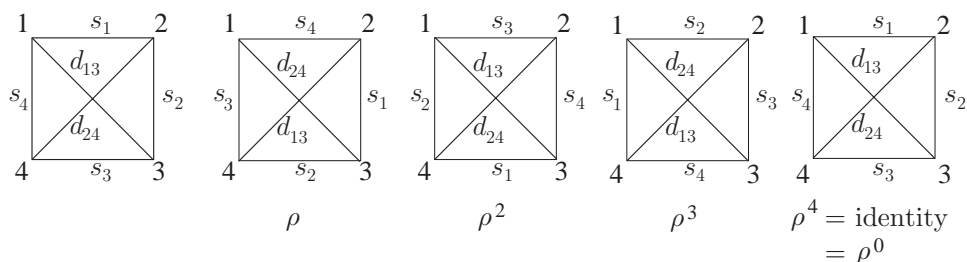
<sup>3</sup>The phrase cyclic group applies in a more general (but similar) situation. Namely the set of all powers of any member of a group is called a cyclic group.

- 275. Write a recurrence for the number  $c(k, n)$  of permutations of  $[k]$  that have exactly  $n$  cycles, including 1-cycles. Use it to write a table of  $c(k, n)$  for  $k$  between 1 and 7 inclusive. Can you find a relationship between  $c(k, n)$  and any of the other families of special numbers such as binomial coefficients, Stirling numbers, Lah numbers, etc. we have studied? If you find such a relationship, prove you are right.
- 276. (Relevant to Appendix C.) A permutation  $\sigma$  is called an **involution** if  $\sigma^2 = \iota$ . When you write down the cycle decomposition of an involution, what is special about the cycles?

## 6.2 Groups Acting on Sets

We defined the rotation group  $C_4$  and the dihedral group  $D_4$  as groups of permutations of the vertices of a square. These permutations represent rigid motions of the square in the plane and in three-dimensional space respectively. The square has geometric features of interest other than its vertices; for example, its diagonals, or its edges. Any geometric motion of the square that returns it to its original location takes each diagonal to a possibly different diagonal, and takes each edge to a possibly different edge. In Figure 6.5 we show the results on the sides and diagonals of the rotations of a square. The rotation group permutes the sides of the square

Figure 6.5: The results on the sides and diagonals of rotating the square



and permutes the diagonals of the square as it rotates the square. Thus we say the rotation group “acts” on the sides and diagonals of the square.

- 277. (a) Write down the two-line notation for the permutation  $\bar{\rho}$  that a 90 degree rotation does to the sides of the square.

- (b) Write down the two-line notation for the permutation  $\overline{\rho^2}$  that a 180 degree rotation does to the sides of the square.
- (c) Is  $\overline{\rho^2} = \overline{\rho} \circ \overline{\rho}$ ? Why or why not?
- (d) Write down the two-line notation for the permutation  $\widehat{\rho}$  that a 90 degree rotation does to the diagonals  $d_{13}$ , and  $d_{24}$  of the square.
- (e) Write down the two-line notation for the permutation  $\widehat{\rho^2}$  that a 180 degree rotation does to the diagonals of the square.
- (f) Is  $\widehat{\rho^2} = \widehat{\rho} \circ \widehat{\rho}$ ? Why or why not? What familiar permutation is  $\widehat{\rho^2}$  in this case?

We have seen that the fact that we have defined a permutation group as the permutations of some specific set doesn't preclude us from thinking of the elements of that group as permuting the elements of some other set as well. In order to keep track of which permutations of which set we are using to define our group and which other set is being permuted as well, we introduce some new language and notation. We are going to say that the group  $D_4$  "acts" on the edges and diagonals of a square and the group  $R$  of permutations of the vertices of a cube that arise from rigid motions of the cube "acts" on the edges, faces, diagonals, etc. of the cube.

- 278. In Figure 6.3 we show a cube with the positions of its vertices and faces labeled. As with motions of the square, we let  $\varphi(x)$  be the label of the place where vertex previously in position  $x$  is now.
- 279. In Problem 261 we wrote in two row notation the permutation  $\rho$  of the vertices that corresponds to rotating the cube 90 degrees around a vertical axis through the faces  $t$  (for top) and  $u$  (for underneath). (We rotated in a right-handed fashion around this axis, meaning that vertex 6 goes to the back and vertex 8 comes to the front.) Write in two row notation the permutation  $\overline{\rho}$  of the faces that corresponds to this member  $\rho$  of  $R$ .
  - (a) In Problem 261 we wrote in two row notation the permutation  $\varphi$  that rotates the cube 120 degrees around the diagonal from vertex 1 to vertex 7 and carries vertex 8 to vertex 6. Write in two row notation the permutation  $\overline{\varphi}$  of the faces that corresponds to this member of  $R$ .
  - (b) In Problem 261 we computed the two row notation for  $\rho \circ \varphi$ . Now compute the two row notation for  $\overline{\rho} \circ \overline{\varphi}$  ( $\overline{\rho}$  was defined in Part 279), and write in two row notation the permutation  $\overline{\rho \circ \varphi}$

of the faces that corresponds to the action of the permutation  $\rho \circ \varphi$  on the faces of the cube (for this question it helps to think geometrically about what motion of the cube is carried out by  $\rho \circ \varphi$ ). What do you observe about  $\overline{\rho \circ \varphi}$  and  $\overline{\rho} \circ \overline{\varphi}$ ?

We say that a permutation group  $G$  **acts** on a set  $S$  if, for each member  $\sigma$  of  $G$  there is a permutation  $\overline{\sigma}$  of  $S$  such that

$$\overline{\sigma \circ \varphi} = \overline{\sigma} \circ \overline{\varphi}$$

for every member  $\sigma$  and  $\varphi$  of  $G$ . In Problem 279b you saw one example of this condition. If we think intuitively of  $\rho$  and  $\varphi$  as motions in space, then following the action of  $\varphi$  by the action of  $\rho$  does give us the action of  $\rho \circ \varphi$ . We can also reason directly with the permutations in the group  $R$  of rigid motions (rotations) of the cube to show that  $R$  acts on the faces of the cube.

- 280. Show that a group  $G$  of permutations of a set  $S$  acts on  $S$  with  $\overline{\varphi} = \varphi$  for all  $\varphi$  in  $G$ .
- 281. The group  $D_4$  is a group of permutations of  $\{1, 2, 3, 4\}$  as in Problem 253. We are going to show in this problem how this group acts on the two-element subsets of  $\{1, 2, 3, 4\}$ . In Problem 286 we will see a natural geometric interpretation of this action. In particular, for each two-element subset  $\{i, j\}$  of  $\{1, 2, 3, 4\}$  and each member  $\sigma$  of  $D_4$  we define  $\overline{\sigma}(\{i, j\}) = \{\sigma(i), \sigma(j)\}$ . Show that with this definition of  $\overline{\sigma}$ , the group  $D_4$  acts on the two-element subsets of  $\{1, 2, 3, 4\}$ .
- 282. Suppose that  $\sigma$  and  $\varphi$  are permutations in the group  $R$  of rigid motions of the cube. We have argued already that each rigid motion sends a face to a face. Thus  $\sigma$  and  $\varphi$  both send the vertices on one face to the vertices on another face. Let  $\{h, i, j, k\}$  be the set of positions of the vertices on a face  $F$ .
  - (a) What are the positions of the vertices of the face  $F'$  that  $F$  is sent to by  $\varphi$ ? (The function  $\varphi$  may appear in your answer.)
  - (b) What are the positions of the vertices of the face  $F''$  that  $F'$  is sent to by  $\sigma$ ?
  - (c) What are the positions of the vertices of the face  $F'''$  that  $F$  is sent to by  $\sigma \circ \varphi$ ?
  - (d) How have you just shown that the group  $R$  acts on the faces?

### 6.2.1 Groups acting on colorings of sets

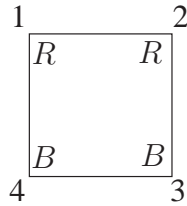
Recall that when you were asked in Problem 44 to find the number of ways to place two red beads and two blue beads at the corners of a square free to move in three-dimensional space, you were not able to apply the quotient principle to answer the question. Instead you had to see that you could divide the set of six lists of two  $R$ s and two  $B$ s into two sets, one of size two in which the  $R$ s and  $B$ s alternated and one of size four in which the two reds (and therefore the two blues) would be side-by-side on the square. Saying that the square is free to move in space is equivalent to saying that two arrangements of beads on the square are equivalent if a member of the dihedral group carries one arrangement to the other. Thus an important ingredient in the analysis of such problems will be how a group can act on colorings of a set of vertices. We can describe the coloring of the square in Figure 6.6 as the function  $f$  with

$$f(1) = R, f(2) = R, f(3) = B, \text{ and } f(4) = B,$$

but it is more compact and turns out to be more suggestive to represent the coloring in Figure 6.6 as the set of ordered pairs

$$(1, R), (2, R), (3, B), (4, B). \quad (6.1)$$

Figure 6.6: The colored square with coloring  $\{(1, R), (2, R), (3, B), (4, B)\}$



This gives us an explicit list of which colors are assigned to which vertex.<sup>4</sup> Then if we rotate the square through 90 degrees, we see that the set of ordered pairs becomes

$$\{(\rho(1), R), (\rho(2), R), (\rho(3), B), (\rho(4), B)\} \quad (6.2)$$

---

<sup>4</sup>The reader who has studied Appendix A will recognize that this set of ordered pairs is the relation of the function  $f$ , but we won't need to make any specific references to the idea of a relation in what follows.



which is the same as

$$\{(2, R), (3, R), (4, B), (1, B)\}.$$

or, in a more natural order,

$$\{(1, B), (2, R), (3, R), (4, B)\}. \quad (6.3)$$

The reordering we did in 6.3 suggests yet another simplification of notation. So long as we know that the first elements of our pairs are labeled by the members of  $[n]$  for some integer  $n$  and we are listing our pairs in increasing order by the first component, we can denote the coloring

$$\{(1, B), (2, R), (3, R), (4, B)\}$$

by  $BRRB$ . In the case where we have numbered the elements of the set  $S$  we are coloring, we will call this list of colors of the elements of  $S$  in order the *standard notation* for the coloring. We will call the ordering used in 6.3 the *standard ordering* of the coloring.

Thus we have three natural ways to represent a coloring of a set as a function, as a set of ordered pairs, and as a list. Different representations are useful for different things. For example, the representation by ordered pairs will provide a natural way to define the action of a group on colorings of a set. Given a coloring as a function  $f$ , we denote the set of ordered pairs

$$\{(x, f(x)) \mid x \in S\},$$

suggestively as  $(S, f)$  for short. We use  $f(1)f(2)\cdots f(n)$  to stand for a particular coloring  $(S, f)$  in the standard notation.

- 283. Suppose now that instead of coloring the vertices of a square, we color its edges. We will use the shorthand 12, 23, 34, and 41 to stand for the edges of the cube between vertex 1 and vertex 2, vertex 2 and vertex 3, and so on. Then a coloring of the edges with 12 red, 23 blue, 34 red and 41 blue can be represented as

$$\{(12, R), (23, B), (34, R), (41, B)\}. \quad (6.4)$$

If  $\rho$  is the rotation through 90 degrees, then we have a permutation  $\bar{\rho}$  acting on the edges. This permutation acts on the colorings to give us a permutation  $\overline{\bar{\rho}}$  of the set of colorings.

- (a) What is  $\overline{\bar{\rho}}$  of the coloring in 6.4?

(b) What is  $\overline{\overline{\rho^2}}$  of the coloring in 6.4?

If  $G$  is a group that acts the set  $S$ , we define the **action of  $G$  on the colorings**  $(S, f)$  by

$$\overline{\overline{(S, f)}} = \overline{\overline{\{(x, f(x)) | x \in S\}}} = \{(\overline{\overline{\sigma}}(x), f(x)) | x \in S\}. \quad (6.5)$$

We have the two bars over  $\sigma$ , because  $\sigma$  is a permutation of one set that gives us a permutation  $\overline{\sigma}$  of a second set, and then  $\overline{\sigma}$  acts to give a permutation  $\overline{\overline{\sigma}}$  of a third set, the set of colorings. For example, suppose we want to analyze colorings of the faces of a cube under the action of the rotation group of the cube as we have defined it on the vertices. Each vertex-permutation  $\sigma$  in the group gives a permutation  $\overline{\sigma}$  of the faces of the cube. Then each permutation  $\overline{\sigma}$  of the faces gives us a permutation  $\overline{\overline{\sigma}}$  of the colorings of the faces.

In the special case that  $G$  is a group of permutations of  $S$  rather than a group acting on  $S$ , Equation 6.5 becomes

$$\overline{\overline{(S, f)}} = \overline{\overline{\{(x, f(x)) | x \in S\}}} = \{(\sigma(x), f(x)) | x \in S\}.$$

In the case where  $G$  is the rotation group of the square acting on the vertices of the square, the example of acting on a coloring by  $\rho$  that we saw in 6.3 is an example of this kind of action. In the standard notation, when we act on a coloring by  $\sigma$ , the color in position  $i$  moves to position  $\sigma(i)$ .

284. Why does the action we have defined on colorings in Equation 6.5 take a coloring to a coloring?
285. Show that if  $G$  is a group of permutations of a set  $S$ , and  $f$  is a coloring function on  $S$ , then the equation

$$\overline{\overline{\{(x, f(x)) | x \in S\}}} = \{(\overline{\overline{\sigma}}(x), f(x)) | x \in S\}$$

defines an action of  $G$  on the colorings  $(S, f)$  of  $S$ .

### 6.2.2 Orbits

- 286. In Problem 281
  - (a) What is the set of two element subsets that you get by computing  $\overline{\overline{\sigma}}(\{1, 2\})$  for all  $\sigma$  in  $D_4$ ?
  - (b) What is the multiset of two-element subsets that you get by computing  $\overline{\overline{\sigma}}(\{1, 2\})$  for all  $\sigma$  in  $D_4$ ?

- (c) What is the set of two-element subsets you get by computing  $\bar{\sigma}(\{1, 3\})$  for all  $\sigma$  in  $D_4$ ?
- (d) What is the multiset of two-element subsets that you get by computing  $\bar{\sigma}(\{1, 3\})$  for all  $\sigma$  in  $D_4$ ?
- (e) Describe the two sets in parts (a) and (c) geometrically in terms of the square.
- 287. This problem uses the notation for permutations in the dihedral group of the square introduced before Problem 257. What is the effect of a 180 degree rotation  $\rho^2$  on the diagonals of a square? What is the effect of the flip  $\varphi_{1|3}$  on the diagonals of a square? How many elements of  $D_4$  send each diagonal to itself? How many elements of  $D_4$  interchange the diagonals of a square?

In Problem 286 you saw that the action of the dihedral group  $D_4$  on two element subsets of  $\{1, 2, 3, 4\}$  seems to split them into two blocks, one with two elements and one with 4. We call these two blocks the “orbits” of  $D_4$  acting on the two element subsets of  $\{1, 2, 3, 4\}$ . More generally, given a group  $G$  acting on a set  $S$ , the **orbit** of  $G$  determined by an element  $x$  of  $S$  is the set

$$\{\bar{\sigma}(x) | \sigma \in G\},$$

and is denoted by  $Gx$ .

288. Suppose a group acts on a set  $S$ . Could an element of  $S$  be in two different orbits? (Say why or why not.)

Problem 288 almost completes the proof of the following theorem.

**Theorem 9** *Suppose a group acts on a set  $S$ . The orbits of  $G$  form a partition of  $S$ .*

It is probably worth pointing out that this theorem tells us that the orbit  $Gx$  is also the orbit  $Gy$  for any element  $y$  of  $Gx$ .

289. Complete the proof of Theorem 9.

Notice that thinking in terms of orbits actually hides some information about the action of our group. When we computed the multiset of all results of acting on  $\{1, 2\}$  with the elements of  $D_4$ , we got an eight-element multiset containing each side twice. When we computed the multiset of all results of acting on  $\{1, 3\}$  with the elements of  $D_4$ , we got an eight-element multiset

containing each diagonal of the square four times. These multisets remind us that we are acting on our two-element sets with an eight-element group. The **multiorbit** of  $G$  determined by an element  $x$  of  $S$  is the multiset

$$\{\bar{\sigma}(x) \mid \sigma \in G\},$$

and is denoted by  $Gx_{\text{multi}}$ .

When we used the quotient principle to count circular seating arrangements or necklaces, we partitioned up a set of lists of people or beads into blocks of equivalent lists. In the case of seating  $n$  people around a round table, what made two lists equivalent was, in retrospect, the action of the rotation group  $C_n$ . In the case of stringing  $n$  beads on a string to make a necklace, what made two lists equivalent was the action of the dihedral group. Thus the blocks of our partitions were orbits of the rotation group or the dihedral group, and we were counting the number of orbits of the group action. In Problem 44, we were not able to apply the quotient principle because we had blocks of different sizes. However, these blocks were still orbits of the action of the group  $D_4$ . And, even though the orbits have different sizes, we expect that each orbit corresponds naturally to a multiorbit and that the multiorbits all have the same size. Thus if we had a version of the quotient rule for a union of multisets, we could hope to use it to count the number of multiorbits.

- 290. (a) Find the orbit and multiorbit of  $D_4$  acting on the coloring

$$\{(1, R), (2, R), (3, B), (4, B)\},$$

or, in standard notation,  $RRBB$ , of the vertices of a square.

- (b) How many group elements map the coloring  $RRBB$  to itself? What is the multiplicity of  $RRBB$  in its multiorbit?
- (c) Find the orbit and multiorbit of  $D_4$  acting on the coloring

$$\{(1, R), (2, B), (3, R), (4, B)\}.$$

- (d) How many elements of the group send the coloring  $RBRB$  to itself? What is the multiplicity of  $RBRB$  in its orbit?

291. (a) If  $G$  is a group, how is the set  $\{\tau\sigma \mid \tau \in G\}$  related to  $G$ ?
- (b) Use this to show that  $y$  is in the multiorbit  $Gx_{\text{multi}}$  if and only if  $Gx_{\text{multi}} = Gy_{\text{multi}}$ .

Problem 291b tells us that, when  $G$  acts on  $S$ , each element  $x$  of  $S$  is in one and only one multiorbit. Since each orbit is a subset of a multiorbit and each element  $x$  of  $S$  is in one and only one orbit, this also tells us there is a bijection between the orbits of  $G$  and the multiorbits of  $G$ , so that we have the same number of orbits as multiorbits.

When a group acts on a set, a group element is said to **fix** an element of  $x \in S$  if  $\bar{\sigma}(x) = x$ . The set of all elements fixing an element  $x$  is denoted by  $\text{Fix}(x)$ .

292. Suppose a group  $G$  acts on a set  $S$ . What is special about the subset  $\text{Fix}(x)$  for an element  $x$  of  $S$ ?
- 293. Suppose a group  $G$  acts on a set  $S$ . What is the relationship of the multiplicity of  $x \in S$  in its multiorbit and the size of  $\text{Fix}(x)$ ?
294. What can you say about the multiplicity of an element  $y$  in the multiorbit  $Gx_{\text{multi}}$ ? Try to use this to get a relationship between the size of an orbit of  $G$  and the size of  $G$ .

We suggested earlier that a quotient principle for multisets might prove useful. The quotient principle came from the sum principle, and we do not have a sum principle for multisets. Such a principle would say that the size of a union of disjoint multisets is the sum of their sizes. We have not yet defined the union of multisets or disjoint multisets, because we haven't needed the ideas until now. We define the *union* of two multisets  $S$  and  $T$  to be the multiset in which the multiplicity of an element  $x$  is the maximum<sup>5</sup> of the multiplicity of  $x$  in  $S$  and its multiplicity in  $T$ . Similarly, the union of a family of multisets is defined by defining the multiplicity of an element  $x$  to be the maximum of its multiplicities in the members of the family. Two multisets are said to be *disjoint* if no element is a member of both, that is, if no element has multiplicity one or more in both. Since the size of a multiset is the sum of the multiplicities of its members, we immediately get the *sum principle for multisets*.

The size of a union of disjoint multisets is the sum of their sizes.

Taking the multisets all to have the same size, we get the *product principle for multisets*.

---

<sup>5</sup>We choose the maximum rather than the sum so that the union of sets is a special case of the union of multisets.

The union of a set of  $m$  disjoint multisets, each of size  $n$  has size  $mn$ .

The *quotient principle for multisets* then follows immediately.

If a  $p$ -element multiset is a union of  $q$  disjoint multisets, each of size  $r$ , then  $q = p/r$ .

- 295. How does the size of the union of the set of multi-orbits of a group  $G$  acting on a set  $S$  relate to the number of multi-orbits and the size of  $G$ ?
- 296. How does the size of the union of the set of multi-orbits of a group  $G$  acting on a set  $S$  relate to the numbers  $|\text{Fix}(x)|$ ?
- 297. In Problems 295 and 296 you computed the size of the union of the set of multi-orbits of a group  $G$  acting on a set  $S$  in two different ways, getting two different expressions which must be equal. Write the equation that says they are equal and solve for the number of multi-orbits, and therefore the number of orbits.

### 6.2.3 The Cauchy-Frobenius-Burnside Theorem

- 298. In Problem 297 you stated and proved a theorem that expresses the number of orbits in terms of the number of group elements fixing each element of  $S$ . It is often easier to find the number of elements fixed by a given group element than to find the number of group elements fixing an element of  $S$ .
  - (a) For this purpose, how does the sum  $\sum_{x \in S} |\text{Fix}(x)|$  relate to the number of ordered pairs  $(\sigma, x)$  (with  $\sigma \in G$  and  $x \in S$ ) such that  $\sigma$  fixes  $x$ ?
  - (b) Let  $\chi(\sigma)$  denote the number of elements of  $S$  fixed by  $\sigma$ . How can the number of ordered pairs  $(\sigma, x)$  (with  $\sigma \in G$  and  $x \in S$ ) such that  $\sigma$  fixes  $x$  be computed from  $\chi(\sigma)$ ?
  - (c) What does this tell you about the number of orbits?
- 299. A second computation of the result of problem 298 can be done as follows.
  - (a) Let  $\chi(\sigma, x) = 1$  if  $\sigma(x) = x$  and let  $\chi(\sigma, x) = 0$  otherwise. Use  $\chi$  to convert the single summation in your answer to Problem 297

into a double summation over elements  $x$  of  $S$  and elements  $\sigma$  of  $G$ .

- (b) Reverse the order of the previous summation in order to convert it into a single sum involving the function  $\chi$  given by

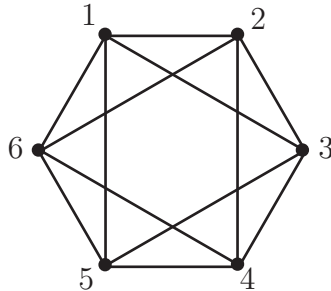
$$\chi(\sigma) = \text{the number of elements of } S \text{ left fixed by } \sigma.$$

In Problem 298 you gave a formula for the number of orbits of a group  $G$  acting on a set  $X$ . This formula was first worked out by Cauchy in the case of the symmetric group, and then for more general groups by Frobenius. In his pioneering book on Group Theory, Burnside used this result as a lemma, and while he attributed the result to Cauchy and Frobenius in the first edition of his book, in later editions, he did not. Later on, other mathematicians who used his book named the result “Burnside’s Lemma,” which is the name by which it is still most commonly known. Let us agree to call this result the Cauchy-Frobenius-Burnside Theorem, or CFB Theorem for short in a compromise between historical accuracy and common usage.

- 300. In how many ways may we string four (identical) red, six (identical) blue, and seven (identical) green beads on a necklace?
- 301. If we have an unlimited supply of identical red beads and identical blue beads, in how many ways may we string 17 of them on a necklace?
- 302. If we have five (identical) red, five (identical) blue, and five (identical) green beads, in how many ways may we string them on a necklace?
- 303. In how many ways may we paint the faces of a cube with six different colors, using all six?
- 304. In how many ways may we paint the faces of a cube with two colors of paint? What if both colors must be used?
- 305. In how many ways may we color the edges of a (regular)  $(2n + 1)$ -gon free to move around in the *plane* (so it cannot be flipped) if we use red  $n$  times and blue  $n + 1$  times? If this is a number you have seen before, identify it.
- \*306. In how many ways may we color the edges of a (regular)  $(2n + 1)$ -gon free to move in *three-dimensional space* so that  $n$  edges are colored red and  $n + 1$  edges are colored blue? Hint: your answer may depend on whether  $n$  is even or odd.

- \*307. (Not unusually hard for someone who has worked on chromatic polynomials.) How many different proper colorings with four colors are there of the vertices of a graph which is a cycle on five vertices? (If we get one coloring by rotating or flipping another one, they aren't really different.)
- \*308. How many different proper colorings with four colors are there of the graph in Figure 6.7? Two graphs are the same if we can redraw one of the graphs, not changing the vertex set or edge set, so that it is identical to the other one. This is equivalent to permuting the vertices in some way so that when we apply the permutation to the endpoints of the edges to get a new edge set, the new edge set is equal to the old one. Such a permutation is called an *automorphism* of the graph. Thus two colorings are different if there is no automorphism of the graph that carries one to the other one.

Figure 6.7: A graph on six vertices.



### 6.3 Pólya-Redfield Enumeration Theory

George Pólya and Robert Redfield independently developed a theory of generating functions that describe the action of a group  $G$  on colorings of a set  $S$  by a set  $T$  when we know the action of  $G$  on  $S$ . Pólya's work on the subject is very accessible in its exposition, and so the subject has become popularly known as Pólya theory, though Pólya-Redfield theory would be a better name. In this section we develop the elements of this theory.

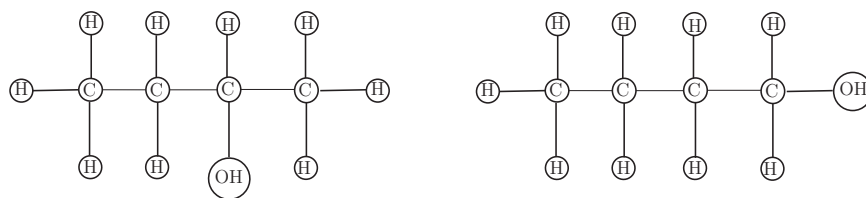
The idea of coloring a set  $S$  has many applications. For example, the set  $S$  might be the positions in a hydrocarbon molecule which are occupied



by hydrogen, and the group could be the group of spatial symmetries of the molecule (that is, the group of permutations of the atoms of the molecule that move the molecule around so that in its final position the molecule cannot be distinguished from the original molecule). The colors could then be radicals (including hydrogen itself) that we could substitute for each hydrogen position in the molecule. Then the number of orbits of colorings is the number of chemically different compounds we could create by using these substitutions.<sup>6</sup>

In Figure 6.8 we show two different ways to substitute the OH radical for a hydrogen atom in the chemical diagram we gave for butane in Chapter 2. We have colored one vertex of degree 1 with the radical OH and the rest with the atom H. There are only two distinct ways to do this, as the OH must either connect to an “end” C or a “middle” C. This shows that there are two different forms, called *isomers* of the compound shown. This compound is known as butyl alcohol.

Figure 6.8: The two different isomers of butyl alcohol.



So think intuitively about some “figure” that has places to be colored. (Think of the faces of a cube, the beads on a necklace, circles at the vertices of an  $n$ -gon, etc.) How can we picture the coloring? If we number the places to be colored, say 1 to  $n$ , then we have a standard way to represent our coloring. For example, if our colors are blue, green and red, then *BBGRRGBG* describes a typical coloring of 8 such places. Unless the places are somehow “naturally” numbered, this idea of a coloring imposes structure that is not

<sup>6</sup>There is a fascinating subtle issue of what makes two molecules different. For example, suppose we have a molecule in the form of a cube, with one atom at each vertex. If we interchange the top and bottom faces of the cube, each atom is still connected to exactly the same atoms as before. However, we cannot achieve this permutation of the vertices by a member of the rotation group of the cube. It could well be that the two versions of the molecule interact with other molecules in different ways, in which case we would consider them chemically different. On the other hand, if the two versions interact with other molecules in the same way, we would have no reason to consider them chemically different. This kind of symmetry is an example of what is called *chirality* in chemistry.

really there. Even if the structure is there, visualizing our colorings in this way doesn't "pull together" any common features of different colorings; we are simply visualizing all possible colorings. We have a group (think of it as symmetries of the figure you are imagining) that acts on the places. That group then acts in a natural way on the colorings of the places and we are interested in orbits of the colorings. Thus we want a picture that pulls together the common features of the colorings in an orbit. One way to pull together similarities of colorings would be to let the letters we are using as pictures of colors commute as we did with our pictures in Chapter 4; then our picture  $BBGRRGBG$  becomes  $B^3G^3R^2$ , so our picture now records simply how many times we use each color. Think about how we defined the action of a group on the colorings of a set on which the group acts. You will see that acting with a group element won't change how many times each color is used; it simply moves colors to different places. Thus the picture we now have of a given coloring is an equally appropriate picture for each coloring in an orbit. One natural question for us to ask is "How many orbits have a given picture?"

- 309. Suppose we draw identical circles at the vertices of a regular hexagon. Suppose we color these circles with two colors, red and blue.
  - (a) In how many ways may we color the set  $\{1, 2, 3, 4, 5, 6\}$  using the colors red and blue?
  - (b) These colorings are partitioned into orbits by the action of the rotation group on the hexagon. Using our standard notation, write down all these orbits and observe how many orbits have each picture, assuming the picture of a coloring is the product of commuting variables representing the colors.
  - (c) Using the picture function of the previous part, write down the picture enumerator for the orbits of colorings of the vertices of a hexagon under the action of the rotation group.

In Problem 309c we saw a picture enumerator for pictures of orbits of the action of a group on colorings. As above, we ask how many orbits of the colorings have any given picture. We can think of a multivariable generating function in which the letters we use to picture individual colors are the variables, and the coefficient of a picture is the number of orbits with that picture. Such a generating function provides an answer to our natural question, and so it is this sort of generating function we will seek. Since the CFB theorem was our primary tool for saying how many orbits we have,

it makes sense to think about whether the CFB theorem has an analog in terms of pictures of orbits.

### 6.3.1 The Orbit-Fixed Point Theorem

- 310. Suppose now we have a group  $G$  acting on a set and we have a picture function on that set with the additional feature that for each orbit of the group, all its elements have the same picture. In this circumstance we define the picture of an orbit or multiorbit to be the picture of any one of its members. The **orbit enumerator**  $\text{Orb}(G, S)$  is the sum of the pictures of the orbits. (Note that this is the same as the sum of the pictures of the *multiorbits*.) The **fixed point enumerator**  $\text{Fix}(G, S)$  is the sum of the pictures of each of the fixed points of each of the elements of  $G$ . We are going to construct a generating function analog of the CFB theorem. The main idea of the proof of the CFB theorem was to try to compute in two different ways the number of elements (i.e. the sum of all the multiplicities of the elements) in the union of all the multiorbits of a group acting on a set. Suppose instead we try to compute the sum of all the pictures of all the elements in the union of the multiorbits of a group acting on a set. By thinking about how this sum relates to  $\text{Orb}(G, S)$  and  $\text{Fix}(G, S)$ , find an analog of the CFB theorem that relates these two enumerators. State and prove this theorem.

We will call the theorem of Problem 310 the **Orbit-Fixed Point Theorem**. In order to apply the Orbit-Fixed Point Theorem, we need some basic facts about picture enumerators.

- 311. Suppose that  $P_1$  and  $P_2$  are picture functions on sets  $S_1$  and  $S_2$  in the sense of Section 4.1.2. Define  $P$  on  $S_1 \times S_2$  by  $P(x_1, x_2) = P_1(x_1)P_2(x_2)$ . How are  $E_{P_1}$ ,  $E_{P_2}$ , and  $E_P$  related? (Hint: you may have already done this problem in another context!)
- 312. Suppose  $P_i$  is a picture function on a set  $S_i$  for  $i = 1, \dots, k$ . We define the picture of a  $k$ -tuple  $(x_1, x_2, \dots, x_k)$  to be the product of the pictures of its elements, i.e.

$$\widehat{P}((x_1, x_2, \dots, x_k)) = \prod_{i=1}^k P_i(x_i).$$

How does the picture enumerator  $E_{\widehat{P}}$  of the set  $S_1 \times S_2 \times \cdots \times S_k$  of all  $k$ -tuples with  $x_i \in S_i$  relate to the picture enumerators of the sets  $S_i$ ? In the special case that  $S_i = S$  for all  $i$  and  $P_i = P$  for all  $i$ , what is  $E_{\widehat{P}}(S^k)$ ?

- 313. Use the Orbit-Fixed Point Theorem to determine the Orbit Enumerator for the colorings, with two colors (red and blue), of six circles placed at the vertices of a hexagon which is free to move in the plane. Compare the coefficients of the resulting polynomial with the various orbits you found in Problem 309.
- 314. Find the generating function (in variables  $R, B$ ) for colorings of the faces of a cube with two colors (red and blue). What does the generating function tell you about the number of ways to color the cube (up to spatial movement) with various combinations of the two colors?

### 6.3.2 The Pólya-Redfield Theorem

Pólya's (and Redfield's) famed enumeration theorem deals with situations such as those in Problems 313 and 314 in which we want a generating function for the set of all colorings a set  $S$  using a set  $T$  of colors, where the picture of a coloring is the product of the multiset of colors it uses. We are again thinking of the colors as variables. The point of the next series of problems is to analyze the solutions to Problems 313 and 314 in order to see what Pólya and Redfield saw (though they didn't see it in this notation or using this terminology).

- 315. In Problem 313 we have four kinds of group elements: the identity (which fixes every coloring), the rotations through 60 or 300 degrees, the rotations through 120 and 240 degrees, and the rotation through 180 degrees. The fixed point enumerator for the rotation group acting on the colorings of the hexagon is by definition the sum of the fixed point enumerators of colorings fixed by the identity, of colorings fixed by 60 or 300 degree rotations, of colorings fixed by 120 or 240 degree rotations, and of colorings fixed by the 180 degree rotation. To the extent that you haven't already done it in an earlier problem, write down each of these enumerators (one for each kind of permutation) individually and factor each one (over the integers) as completely as you can.
- 316. In Problem 314 we have five different kinds of group elements. For each kind of element, to the extent that you haven't already done it

in an earlier problem, write down the fixed point enumerator for the elements of that kind. Factor the enumerators as completely as you can.

- 317. In Problem 315, each “kind” of group element has a “kind” of cycle structure. For example, a rotation through 180 degrees has three cycles of size two. What kind of cycle decomposition does a rotation through 60 or 300 degrees have? What kind of cycle decomposition does a rotation through 120 or 240 degrees have? Discuss the relationship between the cycle structures and the factored enumerators of fixed points of the permutations in Problem 315.

Recall that we said that a group of permutations acts on a set  $S$  if, for each member  $\sigma$  of  $G$  there is a permutation  $\bar{\sigma}$  of  $S$  such that

$$\overline{\sigma \circ \varphi} = \bar{\sigma} \circ \bar{\varphi}$$

for all members  $\sigma$  and  $\varphi$  of  $G$ . Since  $\bar{\sigma}$  is a permutation of  $S$ ,  $\bar{\sigma}$  has a cycle decomposition as a permutation of  $S$  (as well as whatever its cycle decomposition is in the original permutation group  $G$ ).

- 318. In Problem 316, each “kind” of group element has a “kind” of cycle decomposition in the action of the rotation group of the cube on the faces of the cube. For example, a rotation of the cube through 180 degrees around a vertical axis through the centers of the top and bottom faces has two cycles of size two and two cycles of size one. To the extent that you haven’t already done it in an earlier problem, answer the following questions. How many such rotations does the group have? What are the other “kinds” of group elements, and what are their cycle structures? Discuss the relationship between the cycle decomposition and the factored enumerator in Problem 316.
- 319. The usual way of describing the Pólya-Redfield enumeration theorem involves the “cycle indicator” or “cycle index” of a group acting on a set. Suppose we have a group  $G$  acting on a finite set  $S$ . Since each group element  $\sigma$  gives us a permutation  $\bar{\sigma}$  of  $S$ , as such it has a decomposition into disjoint cycles as a permutation of  $S$ . Suppose  $\sigma$  has  $c_1$  cycles of size 1,  $c_2$  cycles of size 2, ...,  $c_n$  cycles of size  $n$ . Then the *cycle monomial* of  $\sigma$  is

$$z(\sigma) = z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}.$$

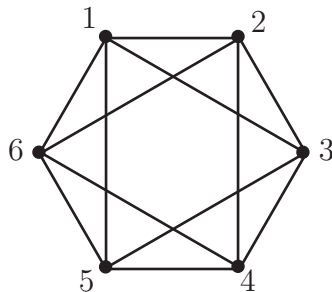
The **cycle indicator** or **cycle index** of  $G$  acting on  $S$  is

$$Z(G, S) = \frac{1}{|G|} \sum_{\sigma: \sigma \in G} z(\sigma).$$

- (a) What is the cycle index for the group  $D_6$  acting on the vertices of a hexagon?
  - (b) What is the cycle index for the group of rotations of the cube acting on the faces of the cube?
- 320. How can you compute the Orbit Enumerator of  $G$  acting on colorings of  $S$  by a finite set  $T$  of colors from the cycle index of  $G$  acting on  $S$ ? (Use  $t$ , thought of as a variable, as the picture of an element  $t$  of  $T$ .) State and prove the relevant theorem! This is Pólya's and Redfield's famous enumeration theorem.
- 321. Suppose we make a necklace by stringing 12 pieces of brightly colored plastic tubing onto a string and fastening the ends of the string together. We have ample supplies of blue, green, red, and yellow tubing available. Give a generating function in which the coefficient of  $B^i G^j R^k Y^h$  is the number of necklaces we can make with  $i$  blues,  $j$  greens,  $k$  reds, and  $h$  yellows. How many terms would this generating function have if you expanded it in terms of powers of  $B$ ,  $G$ ,  $R$ , and  $Y$ ? Does it make sense to do this expansion? How many of these necklaces have 3 blues, 3 greens, 2 reds, and 4 yellows?
- 322. What should we substitute for the variables representing colors in the orbit enumerator of  $G$  acting on the set of colorings of  $S$  by a set  $T$  of colors in order to compute the total number of orbits of  $G$  acting on the set of colorings? What should we substitute into the variables in the cycle index of a group  $G$  acting on a set  $S$  in order to compute the total number of orbits of  $G$  acting on the colorings of  $S$  by a set  $T$ ? Find the number of ways to color the faces of a cube with four colors.
- 323. We have red, green, and blue sticks all of the same length, with a dozen sticks of each color. We are going to make the skeleton of a cube by taking eight identical lumps of modeling clay and pushing three sticks into each lump so that the lumps become the vertices of the cube. (Clearly we won't need all the sticks!) In how many different ways could we make our cube? How many cubes have four edges of each color? How many have two red, four green, and six blue edges?

- 324. How many cubes can we make in Problem 323 if the lumps of modeling clay can be any of four colors?

Figure 6.9: A possible computer network.



- \*325. In Figure 6.9 we see a graph with six vertices. Suppose we have three different kinds of computers that can be placed at the six vertices of the graph to form a network. In how many different ways may the computers be placed? (Two graphs are not different if we can redraw one of the graphs so that it is identical to the other one.) This is equivalent to permuting the vertices in some way so that when we apply the permutation to the endpoints of the edges to get a new edge set, the new edge set is equal to the old one. Such a permutation is called an *automorphism* of the graph. Then two computer placements are the same if there is an automorphism of the graph that carries one to the other.
- 326. Two simple graphs on the set  $[n] = \{1, 2, \dots, n\}$  with edge sets  $E$  and  $E'$  (which we think of as sets of two-element sets for this problem) are said to be *isomorphic* if there is a permutation  $\sigma$  of  $[n]$  which, in its action of two-element sets, carries  $E$  to  $E'$ . We say two graphs are different if they are not isomorphic. Thus the number of different graphs is the number of orbits of the set of all sets of two-element subsets of  $[n]$  under the action of the group  $S_n$ . We can represent an edge set by its characteristic function (as in problem 32). That is, we define

$$\chi_E(\{u, v\}) = \begin{cases} 1 & \text{if } \{u, v\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Thus we can think of the set of graphs as a set of colorings with colors

0 and 1 of the set of all two-element subsets of  $[n]$ . The number of different graphs with vertex set  $[n]$  is thus the number of orbits of this set of colorings under the action of the symmetric group  $S_n$  on the set of two-element subsets of  $[n]$ . Use this to find the number of different graphs on five vertices.

## 6.4 Supplementary Problems

1. Show that a function from  $S$  to  $T$  has an inverse (defined on  $T$ ) if and only if it is a bijection.
2. How many elements are in the dihedral group  $D_3$ ? The symmetric group  $S_3$ ? What can you conclude about  $D_3$  and  $S_3$ ?
3. A tetrahedron is a three-dimensional geometric figure with four vertices, six edges, and four triangular faces. Suppose we start with a tetrahedron in space and consider the set of all permutations of the vertices of the tetrahedron that correspond to moving the tetrahedron in space and returning it to its original location, perhaps with the vertices in different places.
  - (a) Explain why these permutations form a group.
  - (b) What is the size of this group?
  - (c) Write down in two row notation a permutation that is *not* in this group.
4. Find a three-element subgroup of the group  $S_3$ . Can you find a different three-element subgroup of  $S_3$ ?
5. Prove true or demonstrate false with a counterexample: “In a permutation group,  $(\sigma\varphi)^n = \sigma^n\varphi^n$ .”
6. If a group  $G$  acts on a set  $S$ , and if  $\sigma(x) = y$ , is there anything interesting we can say about the subgroups  $\text{Fix}(x)$  and  $\text{Fix}(y)$ ?
7.
  - (a) If a group  $G$  acts on a set  $S$ , does  $\bar{\sigma}(f) = f \circ \sigma$  define a group action on the functions from  $S$  to a set  $T$ ? Why or why not?
  - (b) If a group  $G$  acts on a set  $S$ , does  $\bar{\sigma}(f) = f \circ \sigma^{-1}$  define a group action on the functions from  $S$  to a set  $T$ ? Why or why not?
  - (c) Is either of the possible group actions essentially the same as the action we described on colorings of a set, or is that an entirely different action?



8. Find the number of ways to color the faces of a tetrahedron with two colors.
9. Find the number of ways to color the faces of a tetrahedron with four colors so that each color is used.
10. Find the cycle index of the group of spatial symmetries of the tetrahedron acting on the vertices. Find the cycle index for the same group acting on the faces.
11. Find the generating function for the number of ways to color the faces of the tetrahedron with red, blue, green and yellow.
- 12. Find the generating function for the number of ways to color the faces of a cube with four colors so that all four colors are used.
- 13. How many different graphs are there on six vertices with seven edges?
- 14. Show that if  $H$  is a subgroup of the group  $G$ , then  $H$  acts on  $G$  by  $\bar{\sigma}(\tau) = \sigma \circ \tau$  for all  $\sigma$  in  $H$  and  $\tau$  in  $G$ . What is the size of an orbit of this action? How does the size of a subgroup of a group relate to the size of the group?



# Appendix A

## Relations

### A.1 Relations as Sets of Ordered Pairs

#### A.1.1 The relation of a function

327. Consider the functions from  $S = \{-2, -1, 0, 1, 2\}$  to  $T = \{1, 2, 3, 4, 5\}$  defined by  $f(x) = x + 3$ , and  $g(x) = x^5 - 5x^3 + 5x + 3$ . Write down the set of ordered pairs  $(x, f(x))$  for  $x \in S$  and the set of ordered pairs  $(x, g(x))$  for  $x \in S$ . Are the two functions the same or different?

Problem 327 points out how two functions which appear to be different are actually the same on some domain of interest to us. Most of the time when we are thinking about functions it is fine to think of a function casually as a relationship between two sets. In Problem 327 the set of ordered pairs you wrote down for each function is called the *relation* of the function. When we want to distinguish between the casual and the careful in talking about relationships, our casual term will be “relationship” and our careful term will be “relation.” So *relation* is a technical word in mathematics, and as such it has a technical definition. A *relation* from a set  $S$  to a set  $T$  is a set of ordered pairs whose first elements are in  $S$  and whose second elements are in  $T$ . Another way to say this is that a *relation* from  $S$  to  $T$  is a subset of  $S \times T$ .

A typical way to define a *function*  $f$  from a set  $S$ , called the *domain* of the function, to a set  $T$ , called the *range*, is that  $f$  is a relationship from  $S$  to  $T$  that relates one and only one member of  $T$  to each element of  $X$ . We use  $f(x)$  to stand for the element of  $T$  that is related to the element  $x$  of  $S$ . If we wanted to make our definition more precise, we could substitute the word “relation” for the word “relationship” and we would have a more

precise definition. For our purposes, you can choose whichever definition you prefer. However, in any case, there is a relation associated with each function. As we said above, the relation of a function  $f : S \rightarrow T$  (which is the standard shorthand for “ $f$  is a function from  $S$  to  $T$ ” and is usually read as  $f$  maps  $S$  to  $T$ ) is the set of all ordered pairs  $(x, f(x))$  such that  $x$  is in  $S$ .

328. Here are some questions that will help you get used to the formal idea of a relation and the related formal idea of a function.  $S$  will stand for a finite set of size  $s$  and  $T$  will stand for a finite set of size  $t$ .

- (a) What is the size of the largest relation from  $S$  to  $T$ ?
- (b) What is the size of the smallest relation from  $S$  to  $T$ ?
- (c) The relation of a function  $f : S \rightarrow T$  is the set of all ordered pairs  $(x, f(x))$  with  $x \in S$ . What is the size of the relation of a function from  $S$  to  $T$ ? That is, how many ordered pairs are in the relation of a function from  $S$  to  $T$ ?
- (d) We say  $f$  is a *one-to-one* function or *injection* from  $S$  to  $T$  if each member of  $S$  is related to a *different* element of  $T$ . How many different elements must appear as second elements of the ordered pairs in the relation of a one-to-one function (injection) from  $S$  to  $T$ ?
- (e) A function  $f : S \rightarrow T$  is called an *onto function* or *surjection* if each element of  $T$  is  $f(x)$  for some  $x \in S$ . What is the minimum size that  $S$  can have if there is a surjection from  $S$  to  $T$ ?

329. When  $f$  is a function from  $S$  to  $T$ , the sets  $S$  and  $T$  play a big role in determining whether a function is one-to-one or onto (as defined in Problem 328). For the remainder of this problem, let  $S$  and  $T$  stand for the set of nonnegative real numbers.

- (a) If  $f : S \rightarrow T$  is given by  $f(x) = x^2$ , is  $f$  one-to-one? Is  $f$  onto?
- (b) Now assume for the rest of the problem that  $S'$  is the set of all real numbers and  $g : S' \rightarrow T$  is given by  $g(x) = x^2$ . Is  $g$  one-to-one? Is  $g$  onto?
- (c) Assume for the rest of the problem that  $T'$  is the set of all real numbers and  $h : S \rightarrow T'$  is given by  $h(x) = x^2$ . Is  $h$  one-to-one? Is  $h$  onto?

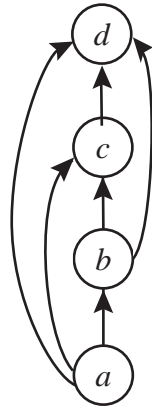
(d) And if the function  $j : S' \rightarrow T'$  is given by  $j(x) = x^2$ , is  $j$  one-to-one? Is  $j$  onto?

330. If  $f : S \rightarrow T$  is a function, we say that  $f$  maps  $x$  to  $y$  as another way to say that  $f(x) = y$ . Suppose  $S = T = \{1, 2, 3\}$ . Give a function from  $S$  to  $T$  that is not onto. Notice that two different members of  $S$  have mapped to the same element of  $T$ . Thus when we say that  $f$  associates one and only one element of  $T$  to each element of  $S$ , it is quite possible that the one and only one element  $f(1)$  that  $f$  maps 1 to is exactly the same as the one and only one element  $f(2)$  that  $f$  maps 2 to.

### A.1.2 Directed graphs

We visualize numerical functions like  $f(x) = x^2$  with their graphs in Cartesian coordinate systems. We will call these kinds of graphs *coordinate graphs* to distinguish them from other kinds of graphs used to visualize relations that are non-numerical. In Figure A.1 we illustrate another kind of graph,

Figure A.1: The alphabet digraph.



a “directed graph” or “digraph” of the “comes before in alphabetical order” relation on the letters  $a$ ,  $b$ ,  $c$ , and  $d$ . To draw a *directed graph* of a relation on a finite<sup>1</sup> set  $S$ , we draw a circle (or dot, if we prefer), which we call a

<sup>1</sup>We could imagine a digraph on an infinite set, but we could never draw all the vertices and edges, so people sometimes speak of digraphs on infinite sets. One just has to be more careful with the definition to make sure it makes sense for infinite sets.

*vertex*, for each element of the set, we usually label the vertex with the set element it corresponds to, and we draw an arrow from the vertex for  $a$  to that for  $b$  if  $a$  is related to  $b$ , that is, if the ordered pair  $(a, b)$  is in our relation. We call such an arrow an *edge* or a *directed edge*. We draw the arrow from  $a$  to  $b$ , for example, because  $a$  comes before  $b$  in alphabetical order. We try to choose the locations where we draw our vertices so that the arrows capture what we are trying to illustrate as well as possible. Sometimes this entails redrawing our directed graph several times until we think the arrows capture the relationship well.

331. Draw the digraph of the “is a proper subset of” relation on the set of subsets of a two element set. How many arrows would you have had to draw if this problem asked you to draw the digraph for the subsets of a three-element set?

We also draw digraphs for relations from a finite set  $S$  to a finite set  $T$ ; we simply draw vertices for the elements of  $S$  (usually in a row) and vertices for the elements of  $T$  (usually in a parallel row) and draw an arrow from  $x$  in  $S$  to  $y$  in  $T$  if  $x$  is related to  $y$ . Notice that instead of referring to the vertex representing  $x$ , we simply referred to  $x$ . This is a common shorthand.

332. Draw the digraph of the relation from the set  $\{A, M, P, S\}$  to the set  $\{\text{Sam, Mary, Pat, Ann, Polly, Sarah}\}$  given by “is the first letter of.”
333. Draw the digraph of the relation from the set  $\{\text{Sam, Mary, Pat, Ann, Polly, Sarah}\}$  to the set  $\{A, M, P, S\}$  given by “has as its first letter.”

### A.1.3 Digraphs of Functions

334. When we draw the digraph of a function  $f$ , we draw an arrow *from* the vertex representing  $x$  *to* the vertex representing  $f(x)$ . One of the relations you considered in Problems 332 and 333 is the relation of a function.
- Which relation is the relation of a function?
  - How does the digraph help you visualize that one relation is a function and the other is not?
335. Digraphs of functions help us to visualize whether or not they are onto or one-to-one. For example, let both  $S$  and  $T$  be the set  $\{-2, -1, 0, 1, 2\}$  and let  $S'$  and  $T'$  be the set  $\{0, 1, 2\}$ . Let  $f(x) = 2 - |x|$ .

- (a) Draw the digraph of the function  $f$  assuming its domain is  $S$  and its range is  $T$ . Use the digraph to explain why or why not this function maps  $S$  onto  $T$ .
- (b) Use the digraph of the previous part to explain whether or not the function is one-to-one.
- (c) Draw the digraph of the function  $f$  assuming its domain is  $S$  and its range is  $T'$ . Use the digraph to explain whether or not the function is onto.
- (d) Use the digraph of the previous part to explain whether or not the function is one-to-one.
- (e) Draw the digraph of the function  $f$  assuming its domain is  $S'$  and its range is  $T'$ . Use the digraph to explain whether the function is onto.
- (f) Use the digraph of the previous part to explain whether the function is one-to-one.
- (g) Suppose that the function  $f$  has domain  $S'$  and range  $T$ . Draw the digraph of  $f$  and use it to explain whether  $f$  is onto.
- (h) Use the digraph of the previous part to explain whether  $f$  is one-to-one.

A one-to-one function from  $S$  onto  $T$  is frequently called a *bijection*, especially in combinatorics. Your work in Problem 335 should show you that a digraph is the digraph of a bijection from  $S$  to  $T$

- if the vertices of the digraph represent the elements of  $S$  and  $T$ ,
- if each vertex representing an element of  $S$  has one and only one arrow leaving it, and
- each vertex representing an element of  $T$  has one and only one arrow entering it.

336. If we reverse all the arrows in the digraph of a bijection  $f$ , we get the digraph of another function  $g$ . What is  $f(g(x))$ ? What is  $g(f(x))$ ?

If  $f$  is a function from  $S$  to  $T$ , if  $g$  is a function from  $T$  to  $S$ , and if  $f(g(x)) = x$  for each  $x$  in  $T$  and  $g(f(x)) = x$  for each  $x$  in  $S$ , then we say that  $g$  is an inverse of  $f$  (and  $f$  is an inverse of  $g$ ).

More generally, if  $f$  is a function from a set  $R$  to a set  $S$ , and  $g$  is a function from  $S$  to  $T$ , then we define a new function  $f \circ g$ , called the *composition* of  $f$  and  $g$ , by  $f \circ g(x) = f(g(x))$ . Composition of functions is a particularly important operation in subjects such as calculus, where we represent a function like  $h(x) = \sqrt{x^2 + 1}$  as the composition of the square root function and the square and add one function in order to use the chain rule to take the derivative of  $h$ .

The function  $\iota$  (the Greek letter iota is pronounced eye-oh-ta) from a set  $S$  to itself, given by the rule  $\iota(x) = x$  for every  $x$  in  $S$ , is called the *identity function* on  $S$ . If  $f$  is a function from  $S$  to  $T$  and  $g$  is a function from  $T$  to  $S$  such that  $g(f(x)) = x$  for every  $x$  in  $S$ , we can express this by saying that  $g \circ f = \iota$ , where  $\iota$  is the identity function of  $S$ . Saying that  $f(g(x)) = x$  is the same as saying that  $f \circ g = \iota$ , where now  $\iota$  stands for the identity function on  $T$ . We use the same letter for the identity function on two different sets when we can use context to tell us on which set the identity function is being defined.

337. If  $f$  is a function from  $S$  to  $T$  and  $g$  is a function from  $T$  to  $S$  such that  $g(f(x)) = x$ , how can we tell from context that  $g \circ f$  is the identity function on  $S$  and not the identity function on  $T$ ?
338. Explain why a function that has an inverse must be a bijection.
339. Is it true that the inverse of a bijection is a bijection?
340. If  $g$  and  $h$  are inverses of  $f$ , when what can we say about  $g$  and  $h$ ?
341. Explain why a bijection must have an inverse.

Since a function with an inverse has exactly one inverse  $g$ , we call  $g$  *the* inverse of  $f$ . From now on, when  $f$  has an inverse, we shall denote its inverse by  $f^{-1}$ . Thus  $f(f^{-1}(x)) = x$  and  $f^{-1}(f(x)) = x$ . Equivalently  $f \circ f^{-1} = \iota$  and  $f^{-1} \circ f = \iota$ .

## A.2 Equivalence Relations

So far we've used relations primarily to talk about functions. There is another kind of relation, called an equivalence relation, that comes up in the counting problems with which we began. In Problem 8 with three distinct flavors, it was probably tempting to say there are 12 flavors for the first pint, 11 for the second, and 10 for the third, so there are  $12 \cdot 11 \cdot 10$  ways to choose the pints of ice cream. However, once the pints have been chosen,



bought, and put into a bag, there is no way to tell which is first, which is second and which is third. What we just counted is lists of three distinct flavors—one-to-one functions from the set  $\{1, 2, 3\}$  in to the set of ice cream flavors. Two of those lists become equivalent once the ice cream purchase is made if they list the same ice cream. In other words, two of those lists become equivalent (are related) if they list same subset of the set of ice cream flavors. To visualize this relation with a digraph, we would need one vertex for each of the  $12 \cdot 11 \cdot 10$  lists. Even with five flavors of ice cream, we would need one vertex for each of  $5 \cdot 4 \cdot 3 = 60$  lists. So for now we will work with the easier to draw question of choosing three pints of ice cream of different flavors from four flavors of ice cream.

342. Suppose we have four flavors of ice cream, V(anilla), C(hocolate), S(trawberry) and P(each). Draw the directed graph whose vertices consist of all lists of three distinct flavors of the ice cream, and whose edges connect two lists if they list the same three flavors. This graph makes it pretty clear in how many “really different” ways we may choose 3 flavors out of four. How many is it?
- 343. Now suppose again we are choosing three distinct flavors of ice cream out of four, but instead of putting scoops in a cone or choosing pints, we are going to have the three scoops arranged symmetrically in a circular dish. Similarly to choosing three pints, we can describe a selection of ice cream in terms of which one goes in the dish first, which one goes in second (say to the right of the first), and which one goes in third (say to the right of the second scoop, which makes it to the left of the first scoop). But again, two of these lists will sometimes be equivalent. Once they are in the dish, we can’t tell which one went in first. However, there is a subtle difference between putting each flavor in its own small dish and putting all three flavors in a circle in a larger dish. Think about what makes the lists of flavors equivalent, and draw the directed graph whose vertices consist of all lists of three of the flavors of ice cream and whose edges connect two lists between which we cannot distinguish as dishes of ice cream. How many dishes of ice cream can we distinguish from one another?
344. Draw the digraph for Problem 37 in the special case where we have four people sitting around the table.

In Problems 342, 343, and 344 (as well as Problems 33c, 37, and 38d) we can begin with a set of lists, and say when two lists are equivalent as

representations of the objects we are trying to count. In particular, in Problems 342, 343, and 344 you drew the directed graph for this relation of equivalence. Technically, you should have had an arrow from each vertex (list) to itself. This is what we mean when we say a relation is *reflexive*. Whenever you had an arrow from one vertex to a second, you had an arrow back to the first. This is what we mean when we say a relation is *symmetric*.

When people sit around a round table, each list is equivalent to itself: if List1 and List2 are identical, then everyone has the same person to the right in both lists (including the first person in the list being to the right of the last person). To see the symmetric property of the equivalence of seating arrangements, if List1 and List2 are different, but everyone has the same person to the right when they sit according to List2 as when they sit according to List1, then everybody better have the same person to the right when they sit according to List1 as when they sit according to List2.

In Problems 342, 343 and 344 there is another property of those relations you may have noticed from the directed graph. Whenever you had an arrow from  $L_1$  to  $L_2$  and an arrow from  $L_2$  to  $L_3$ , then there was an arrow from  $L_1$  to  $L_3$ . This is what we mean when we say a relation is *transitive*. You also undoubtedly noticed how the directed graph divides up into clumps of mutually connected vertices. This is what equivalence relations are all about. Let's be a bit more precise in our description of what it means for a relation to be reflexive, symmetric or transitive.

- If  $R$  is a relation on a set  $X$ , we say  $R$  is *reflexive* if  $(x, x) \in R$  for every  $x \in X$ .
- If  $R$  is a relation on a set  $X$ , we say  $R$  is *symmetric* if  $(x, y)$  is in  $R$  whenever  $(y, x)$  is in  $R$ .
- If  $R$  is a relation on a set  $X$ , we say  $R$  is *transitive* if whenever  $(x, y)$  is in  $R$  and  $(y, z)$  is in  $R$ , then  $(x, z)$  is in  $R$  as well.

Each of the relations of equivalence you worked with in the Problem 342, 343 and 344 had these three properties. Can you visualize the same three properties in the relations of equivalence that you would use in Problems 33c, 37, and 38d? We call a relation an **equivalence relation** if it is reflexive, symmetric and transitive.

After some more examples, we will see how to show that equivalence relations have the kind of clumping property you saw in the directed graphs. In our first example, using the notation  $(a, b) \in R$  to say that  $a$  is related to  $B$  is going to get in the way. It is really more common to write  $aRb$  to

mean that  $a$  is related to  $b$ . For example, if our relation is the less than relation on  $\{1, 2, 3\}$ , you are much more likely to use  $x < y$  than you are  $(x, y) \in <$ , aren't you? The reflexive law then says  $xRx$  for every  $x$  in  $X$ , the symmetric law says that if  $xRy$ , then  $yRx$ , and the transitive law says that if  $xRy$  and  $yRz$ , then  $xRz$ .

345. For the necklace problem, Problem 42, our lists are lists of beads. What makes two lists equivalent for the purpose of describing a necklace? Verify explicitly that this relationship of equivalence is reflexive, symmetric, and transitive.
346. Which of the reflexive, symmetric and transitive properties does the  $<$  relation on the integers have?
347. A relation  $R$  on the set of ordered pairs of positive integers that you learned about in grade school in another notation is the relation that says  $(m, n)$  is related to  $(h, k)$  if  $mk = hn$ . Show that this relation is an equivalence relation. In what context did you learn about this relation in grade school?
348. Another relation that you may have learned about in school, perhaps in the guise of "clock arithmetic," is the relation of equivalence modulo  $n$ . For integers (positive, negative, or zero)  $a$  and  $b$ , we write

$$a \equiv b \pmod{n}$$

to mean that  $a - b$  is an integer multiple of  $n$ , and in this case, we say that  $a$  is *congruent to  $b$  modulo  $n$* . Show that the relation of congruence modulo  $n$  is an equivalence relation.

349. Define a relation on the set of all lists of  $n$  distinct integers chosen from  $\{1, 2, \dots, n\}$ , by saying two lists are related if they have the same elements (though perhaps in a different order) in the first  $k$  places, and the same elements (though perhaps in a different order) in the last  $n - k$  places. Show this relation is an equivalence relation.
350. Suppose that  $R$  is an equivalence relation on a set  $X$  and for each  $x \in X$ , let  $C_x = \{y | y \in X \text{ and } yRx\}$ . If  $C_x$  and  $C_z$  have an element  $y$  in common, what can you conclude about  $C_x$  and  $C_z$  (besides the fact that they have an element in common!)? Be explicit about what property(ies) of equivalence relations justify your answer. Why is every element of  $X$  in some set  $C_x$ ? Be explicit about what property(ies)

of equivalence relations you are using to answer this question. Notice that we might simultaneously denote a set by  $C_x$  and  $C_y$ . Explain why the union of the sets  $C_x$  is  $X$ . Explain why two distinct sets  $C_x$  and  $C_z$  are disjoint. What do these sets have to do with the “clumping” you saw in the digraph of Problem 342 and 343?

In Problem 350 the sets  $C_x$  are called *equivalence classes* of the equivalence relation  $R$ . You have just proved that if  $R$  is an equivalence relation of the set  $X$ , then each element of  $X$  is in exactly one equivalence class of  $R$ . Recall that a *partition* of a set  $X$  is a set of disjoint sets whose union is  $X$ . For example,  $\{1, 3\}$ ,  $\{2, 4, 6\}$ ,  $\{5\}$  is a partition of the set  $\{1, 2, 3, 4, 5, 6\}$ . Thus another way to describe what you proved in Problem 350 is the following:

**Theorem 10** *If  $R$  is an equivalence relation on  $X$ , then the set of equivalence classes of  $R$  is a partition of  $X$ .*

Since a partition of  $S$  is a set of subsets of  $S$ , it is common to call the subsets into which we partition  $S$  the *blocks* of the partition so that we don't find ourselves in the uncomfortable position of referring to a set and not being sure whether it is the set being partitioned or one of the blocks of the partition.

351. In each of Problems 37, 38d, 42, 342, and 343, what does an equivalence class correspond to? (Five answers are expected here.)
352. Given the partition  $\{1, 3\}$ ,  $\{2, 4, 6\}$ ,  $\{5\}$  of the set  $\{1, 2, 3, 4, 5, 6\}$ , define two elements of  $\{1, 2, 3, 4, 5, 6\}$  to be related if they are in the same part of the partition. That is, define 1 to be related to 3 (and 1 and 3 each related to itself), define 2 and 4, 2 and 6, and 4 and 6 to be related (and each of 2, 4, and 6 to be related to itself), and define 5 to be related to itself. Show that this relation is an equivalence relation.
353. Suppose  $P = \{S_1, S_2, S_3, \dots, S_k\}$  is a partition of  $S$ . Define two elements of  $S$  to be related if they are in the same set  $S_i$ , and otherwise not to be related. Show that this relation is an equivalence relation. Show that the equivalence classes of the equivalence relation are the sets  $S_i$ .

In Problem 353 you just proved that each partition of a set gives rise to an equivalence relation whose classes are just the parts of the partition. Thus in Problem 350 and Problem 353 you proved the following Theorem.

**Theorem 11** *A relation  $R$  is an equivalence relation on a set  $S$  if and only if  $S$  may be partitioned into sets  $S_1, S_2, \dots, S_n$  in such a way that  $x$  and  $y$  are related by  $R$  if and only if they are in the same block  $S_i$  of the partition.*

In Problems 342, 343, 37 and 42 what we were doing in each case was counting equivalence classes of an equivalence relation. There was a special structure to the problems that made this somewhat easier to do. For example, in 342, we had  $4 \cdot 3 \cdot 2 = 24$  lists of three distinct flavors chosen from V, C, S, and P. Each list was equivalent to  $3 \cdot 2 \cdot 1 = 3! = 6$  lists, including itself, from the point of view of serving 3 small dishes of ice cream. The order in which we selected the three flavors was unimportant. Thus the set of all  $4 \cdot 3 \cdot 2$  lists was a union of some number  $n$  of equivalence classes, each of size 6. By the product principle, if we have a union of  $n$  disjoint sets, each of size 6, the union has  $6n$  elements. But we already knew that the union was the set of all 24 lists of three distinct letters chosen from our four letters. Thus we have  $6n = 24$ , or  $n = 4$  equivalence classes.

In Problem 343 there is a subtle change. In the language we adopted for seating people around a round table, if we choose the flavors V, C, and S, and arrange them in the dish with C to the right of V and S to the right of C, then the scoops are in different relative positions than if we arrange them instead with S to the right of V and C to the right of S. Thus the order in which the scoops go into the dish is somewhat important—somewhat, because putting in V first, then C to its right and S to its right is the same as putting in S first, then V to its right and C to its right. In this case, each list of three flavors is equivalent to only three lists, including itself, and so if there are  $n$  equivalence classes, we have  $3n = 24$ , so there are  $24/3 = 8$  equivalence classes.

354. If we have an equivalence relation that divides a set with  $k$  elements up into equivalence classes each of size  $m$ , what is the number  $n$  of equivalence classes? Explain why.
355. In Problem 349, what is the number of equivalence classes? Explain in words the relationship between this problem and the Problem 38d.
356. Describe explicitly what makes two lists of beads equivalent in Problem 42 and how Problem 354 can be used to compute the number of different necklaces.
357. What are the equivalence classes (write them out as sets of lists) in Problem 44, and why can't we use Problem 354 to compute the number of equivalence classes?

In Problem 354 you proved our next theorem. In Chapter 1 (Problem 41) we discovered and stated this theorem in the context of partitions and called it the *Quotient Principle*.

**Theorem 12** *If an equivalence relation on a set  $S$  size  $k$  has  $n$  equivalence classes each of size  $m$ , then the number of equivalence classes is  $k/m$ .*

## Appendix B

# Mathematical Induction

### B.1 The Principle of Mathematical Induction

#### B.1.1 The ideas behind mathematical induction

There is a variant of one of the bijections we used to prove the Pascal Equation that comes up in counting the subsets of a set. In the next problem it will help us compute the total number of subsets of a set, regardless of their size. Our main goal in this problem, however, is to introduce some ideas that will lead us to one of the most powerful proof techniques in combinatorics (and many other branches of mathematics), the principle of mathematical induction.

358. (a) Write down a list of the subsets of  $\{1, 2\}$ . Don't forget the empty set! Group the sets containing 2 separately from the others.
- (b) Write down a list of the subsets of  $\{1, 2, 3\}$ . Group the sets containing 3 separately from the others.
- (c) Look for a natural way to match up the subsets containing 2 in Part (a) with those not containing 2. Look for a way to match up the subsets containing 3 in Part (b) containing 3 with those not containing 3.
- (d) On the basis of the previous part, you should be able to find a bijection between the collection of subsets of  $\{1, 2, \dots, n\}$  containing  $n$  and those not containing  $n$ . (If you are having difficulty figuring out the bijection, try rethinking Parts (a) and (b), perhaps by doing a similar exercise with the set  $\{1, 2, 3, 4\}$ .) Describe the bijection (unless you are very familiar with the notation of

sets, it is probably easier to describe the function in words rather than symbols) and explain why it is a bijection. Explain why the number of subsets of  $\{1, 2, \dots, n\}$  containing  $n$  equals the number of subsets of  $\{1, 2, \dots, n - 1\}$ .

- (e) Parts (a) and (b) suggest strongly that the number of subsets of a  $n$ -element set is  $2^n$ . In particular, the empty set has  $2^0$  subsets, a one-element set has  $2^1$  subsets, itself and the empty set, and in Parts a and b we saw that two-element and three-element sets have  $2^2$  and  $2^3$  subsets respectively. So there are certainly some values of  $n$  for which an  $n$ -element set has  $2^n$  subsets. One way to prove that an  $n$ -element set has  $2^n$  subsets for all values of  $n$  is to argue by contradiction. For this purpose, suppose there is a nonnegative integer  $n$  such that an  $n$ -element set doesn't have exactly  $2^n$  subsets. In that case there may be more than one such  $n$ . Choose  $k$  to be the smallest such  $n$ . Notice that  $k - 1$  is still a positive integer, because  $k$  can't be 0, 1, 2, or 3. Since  $k$  was the smallest value of  $n$  we could choose to make the statement "An  $n$ -element set has  $2^n$  subsets" false, what do you know about the number of subsets of a  $(k - 1)$ -element set? What do you know about the number of subsets of the  $k$ -element set  $\{1, 2, \dots, k\}$  that don't contain  $k$ ? What do you know about the number of subsets of  $\{1, 2, \dots, k\}$  that do contain  $k$ ? What does the sum principle tell you about the number of subsets of  $\{1, 2, \dots, k\}$ ? Notice that this contradicts the way in which we chose  $k$ , and the only assumption that went into our choice of  $k$  was that "there is a nonnegative integer  $n$  such that an  $n$ -element set doesn't have exactly  $2^n$  subsets." Since this assumption has led us to a contradiction, it must be false. What can you now conclude about the statement "for every nonnegative integer  $n$ , an  $n$ -element set has exactly  $2^n$  subsets?"

359. The expression

$$1 + 3 + 5 + \dots + 2n - 1$$

is the sum of the first  $n$  odd integers (notice that the  $n$ th odd integer is  $2n - 1$ ). Experiment a bit with the sum for the first few positive integers and guess its value in terms of  $n$ . Now apply the technique of Problem 358 to prove that you are right.

In Problems 358 and 359 our proofs had several distinct elements. We had a statement involving an integer  $n$ . We knew the statement was true



for the first few nonnegative integers in Problem 358 and for the first few positive integers in Problem 359. We wanted to prove that the statement was true for all nonnegative integers in Problem 358 and for all positive integers in Problem 359. In both cases we used the method of proof by contradiction; for that purpose we assumed that there was a value of  $n$  for which our formula wasn't true. We then chose  $k$  to be the smallest value of  $n$  for which our formula wasn't true. This meant that when  $n$  was  $k - 1$ , our formula was true, (or else that  $k - 1$  wasn't a nonnegative integer in Problem 358 or that  $k - 1$  wasn't a positive integer in Problem 359). What we did next was the crux of the proof. We showed that the truth of our statement for  $n = k - 1$  implied the truth of our statement for  $n = k$ . This gave us a contradiction to the assumption that there was an  $n$  that made the statement false. In fact, we will see that we can bypass entirely the use of proof by contradiction. We used it to help you discover the central ideas of the technique of proof by mathematical induction.

The central core of mathematical induction is the proof that the truth of a statement about the integer  $n$  for  $n = k - 1$  implies the truth of the statement for  $n = k$ . For example, once we know that a set of size 0 has  $2^0$  subsets, if we have proved our implication, we can then conclude that a set of size 1 has  $2^1$  subsets, from which we can conclude that a set of size 2 has  $2^2$  subsets, from which we can conclude that a set of size 3 has  $2^3$  subsets, and so on up to a set of size  $n$  having  $2^n$  subsets for any nonnegative integer  $n$  we choose. In other words, although it was the idea of proof by contradiction that led us to think about such an implication, we can now do without the contradiction at all. What we need to prove a statement about  $n$  by this method is a place to start, that is a value  $b$  of  $n$  for which we know the statement to be true, and then a proof that the truth of our statement for  $n = k - 1$  implies the truth of the statement for  $n = k$  whenever  $k > b$ .

### B.1.2 Mathematical induction

The **principle of mathematical induction** states that

In order to prove a statement about an integer  $n$ , if we can

1. Prove the statement when  $n = b$ , for some fixed integer  $b$
2. Show that the truth of the statement for  $n = k - 1$  implies the truth of the statement for  $n = k$  whenever  $k > b$ ,

then we can conclude the statement is true for all integers  $n \geq b$ .

As an example, let us return to Problem 358. The statement we wish to prove is the statement that “A set of size  $n$  has  $2^n$  subsets.”

Our statement is true when  $n = 0$ , because a set of size 0 is the empty set and the empty set has  $1 = 2^0$  subsets. (This step of our proof is called a *base step*.)

Now suppose that  $k > 0$  and every set with  $k - 1$  elements has  $2^{k-1}$  subsets. Suppose  $S = \{a_1, a_2, \dots, a_k\}$  is a set with  $k$  elements. We partition the subsets of  $S$  into two blocks. Block  $B_1$  consists of the subsets that do not contain  $a_n$  and block  $B_2$  consists of the subsets that do contain  $a_n$ . Each set in  $B_1$  is a subset of  $\{a_1, a_2, \dots, a_{k-1}\}$ , and each subset of  $\{a_1, a_2, \dots, a_{k-1}\}$  is in  $B_1$ . Thus  $B_1$  is the set of all subsets of  $\{a_1, a_2, \dots, a_{k-1}\}$ . Therefore by our assumption in the first sentence of this paragraph, the size of  $B_1$  is  $2^{k-1}$ . Consider the function from  $B_2$  to  $B_1$  which takes a subset of  $S$  including  $a_n$  and removes  $a_n$  from it. This function is defined on  $B_2$ , because every set in  $B_2$  contains  $a_n$ . This function is onto, because if  $T$  is a set in  $B_1$ , then  $T \cup \{a_k\}$  is a set in  $B_2$  which the function sends to  $T$ . This function is one-to-one because if  $V$  and  $W$  are two different sets in  $B_2$ , then removing  $a_k$  from them gives two different sets in  $B_1$ . Thus we have a bijection between  $B_1$  and  $B_2$ , so  $B_1$  and  $B_2$  have the same size. Therefore by the sum principle the size of  $B_1 \cup B_2$  is  $2^{k-1} + 2^{k-1} = 2^k$ . Therefore  $S$  has  $2^k$  subsets. This shows that if a set of size  $k - 1$  has  $2^{k-1}$  subsets, then a set of size  $k$  has  $2^k$  subsets. Therefore by the principle of mathematical induction, a set of size  $n$  has  $2^n$  subsets for every nonnegative integer  $n$ .

The first sentence of the last paragraph is called the *inductive hypothesis*. In an inductive proof we always make an inductive hypothesis as part of proving that the truth of our statement when  $n = k - 1$  implies the truth of our statement when  $n = k$ . The last paragraph itself is called the *inductive step* of our proof. In an inductive step we derive the statement for  $n = k$  from the statement for  $n = k - 1$ , thus proving that the truth of our statement when  $n = k - 1$  implies the truth of our statement when  $n = k$ . The last sentence in the last paragraph is called the *inductive conclusion*. All inductive proofs should have a base step, an inductive hypothesis, an inductive step, and an inductive conclusion.

There are a couple details worth noticing. First, in this problem, our base step was the case  $n = 0$ , or in other words, we had  $b = 0$ . However,

in other proofs,  $b$  could be any integer, positive, negative, or 0. Second, our proof that the truth of our statement for  $n = k - 1$  implies the truth of our statement for  $n = k$  required that  $k$  be at least 1, so that there would be an element  $a_k$  we could take away in order to describe our bijection. However, condition (2) of the principle of mathematical induction only requires that we be able to prove the implication for  $k > 0$ , so we were allowed to assume  $k > 0$ .

360. Use mathematical induction to prove your formula from Problem 359.

### B.1.3 Proving algebraic statements by induction

361. Use mathematical induction to prove the well-known formula that for all positive integers  $n$ ,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

362. Experiment with various values of  $n$  in the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n \cdot (n+1)} = \sum_{i=1}^n \frac{1}{i \cdot (i+1)}.$$

Guess a formula for this sum and prove your guess is correct by induction.

363. For large values of  $n$ , which is larger,  $n^2$  or  $2^n$ ? Use mathematical induction to prove that you are correct.

364. What is wrong with the following attempt at an inductive proof that all integers in any consecutive set of  $n$  integers are equal for every positive integer  $n$ ? For an arbitrary integer  $i$ , all integers from  $i$  to  $i$  are equal, so our statement is true when  $n = 1$ . Now suppose  $k > 1$  and all integers in any consecutive set of  $k - 1$  integers are equal. Let  $S$  be a set of  $k$  consecutive integers. By the inductive hypothesis, the first  $k - 1$  elements of  $S$  are equal and the last  $k - 1$  elements of  $S$  are equal. Therefore all the elements in the set  $S$  are equal. Thus by the principle of mathematical induction, for every positive  $n$ , every  $n$  consecutive integers are equal.

## B.2 Strong Induction

One way of looking at the principle of mathematical induction is that it tells us that if we know the “first” case of a theorem and we can derive each other case of the theorem from a smaller case, then the theorem is true in all cases. However, the particular way in which we stated the theorem is rather restrictive in that it requires us to derive each case from the immediately preceding case. This restriction is not necessary, and removing it leads us to a more general statement of the principle of mathematical induction which people often call the **strong principle of mathematical induction**. It states:

In order to prove a statement about an integer  $n$  if we can

1. prove our statement when  $n = b$  and
2. prove that the statements we get with  $n = b, n = b + 1, \dots, n = k - 1$  imply the statement with  $n = k$ ,

then our statement is true for all integers  $n \geq b$ .

365. What postage do you think we can make with five and six cent stamps? Do you think that there is a number  $N$  such that if  $n \geq N$ , then we can make  $n$  cents worth of postage?

You probably see that we can make  $n$  cents worth of postage as long as  $n$  is at least 20. However, you didn't try to make 26 cents in postage by working with 25 cents; rather you saw that you could get 20 cents and then add six cents to that to get 26 cents. Thus if we want to prove by induction that we are right that if  $n \geq 20$ , then we can make  $n$  cents worth of postage, we are going to have to use the strong version of the principle of mathematical induction.

We know that we can make 20 cents with four five-cent stamps. Now we let  $k$  be a number greater than 20, and assume that it is possible to make any amount between 20 and  $k - 1$  cents in postage with five and six cent stamps. Now if  $k$  is less than 25, it is 21, 22, 23, or 24. We can make 21 with three fives and one six. We can make 22 with two fives and two sixes, 23 with one five and three sixes, and 24 with four sixes. Otherwise  $k - 5$  is between 20 and  $k - 1$  (inclusive) and so by our inductive hypothesis, we know that  $k - 5$  cents can be made with five and six cent stamps, so with one more five cent stamp, so can  $k$  cents. Thus by the (strong) principle of mathematical induction, we can make  $n$  cents in stamps with five and six cent stamps for each  $n \geq 20$ .

Some people might say that we really had five base cases,  $n = 20, 21, 22, 23,$  and  $24,$  in the proof above and once we had proved those five consecutive base cases, then we could reduce any other case to one of these base cases by successively subtracting 5. That is an appropriate way to look at the proof. A logician would say that it is also the case that, for example, by proving we could make 22 cents, we also proved that if we can make 20 cents and 21 cents in stamps, then we could also make 22 cents. We just didn't bother to use the assumption that we could make 20 cents and 21 cents! So long as one point of view or the other satisfies you, you are ready to use this kind of argument in proofs.

366. A number greater than one is called prime if it has no factors other than itself and one. Show that each positive number is either a power of a prime (remember what  $p^0$  and  $p^1$  are) or a product of powers of prime numbers.
367. Show that the number of prime factors of a positive number  $n \geq 2$  is less than or equal to  $\log_2 n$ . (If a prime occurs to the  $k$ th power in a factorization of  $n$ , you can consider that power as  $k$  prime factors.) (There is a way to do this by induction and a way to do it without induction. It would be ideal to find both ways.)
368. One of the most powerful statements in elementary number theory is Euclid's Division Theorem<sup>1</sup>. This states that if  $m$  and  $n$  are positive integers, then there are nonnegative integers  $q$  and  $r$  with  $0 \leq r < n$ , such that  $m = nq + r$ . The number  $q$  is called the quotient and the number  $r$  is called the remainder. In computer science it is common to denote  $r$  by  $m \bmod n$ . In elementary school you learned how to use long division to find  $q$  and  $r$ . However, it is unlikely that anyone ever proved for you that for any pair of positive integers,  $m$  and  $n$ , there is such a pair of nonnegative numbers  $q$  and  $r$ . You now have the tools needed to prove this. Do so.

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<sup>1</sup>In a curious twist of language, mathematicians have long called this result Euclid's Algorithm. However as computer science has grown in importance, the word *algorithm* has gotten a more precise definition: an algorithm is now a method to do something. There is a method (in fact there are more than one) to get the  $q$  and  $r$  that Euclid's Division Theorem gives us, and computer scientists would call these methods algorithms. Your author has chosen to break with mathematical tradition and restrict his use of the word algorithm to the more precise interpretation as a computer scientist probably would. We aren't giving a method here, so this is why the name used here is "Euclid's Division Theorem."



## Appendix C

# Exponential Generating Functions

### C.1 Indicator Functions

When we introduced the idea of a generating function, we said that the formal sum

$$\sum_{i=0}^{\infty} a_i x^i$$

may be thought of as a convenient way to keep track of the sequence  $a_i$ . We then did quite a few examples that showed how combinatorial properties of arrangements counted by the coefficients in a generating function could be mirrored by algebraic properties of the generating functions themselves. The monomials  $x^i$  are called *indicator polynomials*. (They indicate the position of the coefficient  $a_i$ .) One example of a generating function is given by

$$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i.$$

Thus we say that  $(1+x)^n$  is the generating function for the binomial coefficients  $\binom{n}{i}$ . The notation tells us that we are assuming that only  $i$  varies in the sum on the right, but that the equation holds for each fixed integer  $n$ . This is implicit when we say that  $(1+x)^n$  is the generating function for  $\binom{n}{i}$ , because we haven't written  $i$  anywhere in  $(1+x)^n$ , so it is free to vary.

Another example of a generating function is given by

$$x^n = \sum_{i=0}^{\infty} s(n, i) x^i.$$

Thus we say that  $x^n$  is the generating function for the Stirling numbers of the first kind,  $s(n, i)$ . There is a similar equation for Stirling numbers of the second kind, namely

$$x^n = \sum_{i=0}^{\infty} S(n, i)x^i.$$

However, with our previous definition of generating functions, this equation would not give a generating function for the Stirling numbers of the second kind, because  $S(n, i)$  is not the coefficient of  $x^i$ . If we were willing to consider the falling factorial powers  $x^{\underline{i}}$  as indicator polynomials, then we could say that  $x^n$  is the generating function for the numbers  $S(n, i)$  relative to these indicator polynomials. This suggests that perhaps different sorts of indicator polynomials go naturally with different sequences of numbers.

The binomial theorem gives us yet another example.

- 369. Write  $(1 + x)^n$  as a sum of multiples of  $\frac{x^i}{i!}$  rather than as a sum of multiples of  $x^i$ .

This example suggests that we could say that  $(1 + x)^n$  is the generating function for the falling factorial powers  $n^{\underline{i}}$  relative to the indicator polynomials  $\frac{x^i}{i!}$ . In general, a sequence of polynomials is called a family of **indicator polynomials** if there is one polynomial of each nonnegative integer degree in the sequence. Those familiar with linear algebra will recognize that this says that a family of indicator polynomials forms a basis for the vector space of polynomials. This means that each polynomial can be expressed as a sum of numerical multiples of indicator polynomials in one and only one way. One could use the language of linear algebra to define indicator polynomials in an even more general way, but a definition in such generality would not be useful to us at this point.

## C.2 Exponential Generating Functions

We say that the expression  $\sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$  is the **exponential generating function** for the sequence  $a_i$ . It is standard to use **EGF** as a shorthand for exponential generating function. In this context we call the generating function  $\sum_{i=0}^n a_i x^i$  that we originally studied the **ordinary generating function** for the sequence  $a_i$ . You can see why we use the term exponential generating function by thinking about the exponential generating function (EGF) for the all ones sequence,

$$\sum_{i=0}^{\infty} 1 \frac{x^i}{i!} = \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x,$$



which we also denote by  $\exp(x)$ . Recall from calculus that the usual definition of  $e^x$  or  $\exp(x)$  involves limits at least implicitly. We work our way around that by defining  $e^x$  to be the power series  $\sum_{i=0}^{\infty} \frac{x^i}{i!}$ .

- 370. Find the EGF (exponential generating function) for the sequence  $a_n = 2^n$ . What does this say about the EGF for the number of subsets of an  $n$ -element set?
- 371. Find the EGF (exponential generating function) for the number of ways to paint the  $n$  streetlight poles that run along the north side of Main Street in Anytown, USA using five colors.
- 372. For what sequence is  $\frac{e^x - e^{-x}}{2} = \cosh x$  the EGF (exponential generating function)?
- 373. For what sequence is  $\ln\left(\frac{1}{1-x}\right)$  the EGF? (The notation  $\ln(y)$  stands for the natural logarithm of  $y$ . People often write  $\log(y)$  instead.) Hint: Think of the definition of the logarithm as an integral, and don't worry at this stage whether or not the usual laws of calculus apply, just use them as if they do! We will then define  $\ln(1-x)$  to be the power series you get.<sup>1</sup>
- 374. What is the EGF for the number of permutations of an  $n$ -element set?
- 375. What is the EGF for the number of ways to arrange  $n$  people around a round table? Notice that we may think of this as the EGF for the number of permutations on  $n$  elements that are cycles.
- 376. What is the EGF  $\sum_{n=0}^{\infty} p_{2n} \frac{x^{2n}}{(2n)!}$  for the number of ways  $p_{2n}$  to pair up  $2n$  people to play a total of  $n$  tennis matches (as in Problems 12a and 43)?

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<sup>1</sup>It is possible to define the derivatives and integrals of power series by the formulas

$$\frac{d}{dx} \sum_{i=0}^{\infty} b_i x^i = \sum_{i=1}^{\infty} i b_i x^{i-1}$$

and

$$\int_0^x \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} \frac{b_i}{i+1} x^{i+1}$$

rather than by using the limit definitions from calculus. It is then possible to prove that the sum rule, product rule, etc. apply. (There is a little technicality involving the meaning of composition for power series that turns into a technicality involving the chain rule, but it needn't concern us at this time.)

- 377. What is the EGF for the sequence  $0, 1, 2, 3, \dots$ ? You may think of this as the EGF for the number of ways to select one element from an  $n$  element set. What is the EGF for the number of ways to select two elements from an  $n$ -element set?
- 378. What is the EGF for the sequence  $1, 1, \dots, 1, \dots$ ? Notice that we may think of this as the EGF for the number of identity permutations on an  $n$ -element set, which is the same as the number of permutations of  $n$  elements whose cycle decomposition consists entirely of 1-cycles, or as the EGF for the number of ways to select an  $n$ -element set (or, if you prefer, an empty set) from an  $n$ -element set. As you may have guessed, there are many other combinatorial interpretations we could give to this EGF.
- 379. What is the EGF for the number of ways to select  $n$  distinct elements from a one-element set? What is the EGF for the number of ways to select a positive number  $n$  of distinct elements from a one-element set? Hint: When you get the answer you will either say “of course,” or “this is a silly problem.”
- 380. What is the EGF for the number of partitions of a  $k$ -element set into exactly one block? (Hint: is there a partition of the empty set into exactly one block?)
- 381. What is the EGF for the number of ways to arrange  $k$  books on one shelf (assuming they all fit)? What is the EGF for the number of ways to arrange  $k$  books on a fixed number  $n$  of shelves, assuming that all the books can fit on any one shelf? (Remember Problem 120.)

### C.3 Applications to Recurrences.

We saw that ordinary generating functions often play a role in solving recurrence relations. We found them most useful in the constant coefficient case. Exponential generating functions are useful in solving recurrence relations where the coefficients involve simple functions of  $n$ , because the  $n!$  in the denominator can cancel out factors of  $n$  in the numerator.

- 382. Consider the recurrence  $a_n = na_{n-1} + n(n-1)$ . Multiply both sides by  $\frac{x^n}{n!}$ , and sum from  $n = 2$  to  $\infty$ . (Why do we sum from  $n = 2$  to infinity instead of from  $n = 1$  or  $n = 0$ ?) Letting  $y = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$ , show that the left-hand side of the equation is  $y - a_0 - a_1x$ . Express the

right hand side in terms of  $y$ ,  $x$ , and  $e^x$ . Solve the resulting equation for  $y$  and use the result to get an equation for  $a_n$ . (A finite summation is acceptable in your answer for  $a_n$ .)

- · 383. The telephone company in a city has  $n$  subscribers. Assume a telephone call involves exactly two subscribers (that is, there are no calls to outside the network and no conference calls), and that the configuration of the telephone network is determined by which pairs of subscribers are talking. Notice that we may think of a configuration of the telephone network as a permutation whose cycle decomposition consists entirely of one-cycles and two-cycles, that is, we may think of a configuration as an involution in the symmetric group  $S_n$ .
- Give a recurrence for the number  $c_n$  of configurations of the network. (Hint: Person  $n$  is either on the phone or not.)
  - What are  $c_0$  and  $c_1$ ?
  - What are  $c_2$  through  $c_6$ ?
- · 384. Recall that a *derangement* of  $[n]$  is a permutation of  $[n]$  that has no fixed points, or equivalently is a way to pass out  $n$  hats to their  $n$  different owners so that nobody gets the correct hat. Use  $d_n$  to stand for the number of derangements of  $[n]$ . We can think of a derangement of  $[n]$  as a list of 1 through  $n$  so that  $i$  is not in the  $i$ th place for any  $n$ . Thus in a derangement, some number  $k$  different from  $n$  is in position  $n$ . Consider two cases: either  $n$  is in position  $k$  or it is not. Notice that in the second case, if we erase position  $n$  and replace  $n$  by  $k$ , we get a derangement of  $[n - 1]$ . Based on these two cases, find a recurrence for  $d_n$ . What is  $d_1$ ? What is  $d_2$ ? What is  $d_0$ ? What are  $d_3$  through  $d_6$ ?

### C.3.1 Using calculus with exponential generating functions

- · 385. Your recurrence in Problem 383 should be a second order recurrence.
- Assuming that the left hand side is  $c_n$  and the right hand side involves  $c_{n-1}$  and  $c_{n-2}$ , decide on an appropriate power of  $x$  divided by an appropriate factorial by which to multiply both sides of the recurrence. Using the fact that the derivative of  $\frac{x^n}{n!}$  is  $\frac{x^{n-1}}{(n-1)!}$ , write down a differential equation for the EGF  $T(x) = \sum_{i=0}^{\infty} c_i \frac{x^i}{i!}$ . Note

that it makes sense to substitute 0 for  $x$  in  $T(x)$ . What is  $T(0)$ ? Solve your differential equation to find an equation for  $T(x)$ .

(b) Use your EGF to compute a formula for  $c_n$ .

→ 386. Your recurrence in Problem 384 should be a second order recurrence.

(a) Assuming that the left-hand side is  $d_n$  and the right hand side involves  $d_{n-1}$  and  $d_{n-2}$ , decide on an appropriate power of  $x$  divided by an appropriate factorial by which to multiply both sides of the recurrence. Using the fact that the derivative of  $\frac{x^n}{n!}$  is  $\frac{x^{n-1}}{(n-1)!}$ , write down a differential equation for the EGF  $D(x) = \sum_{i=0}^{\infty} d_i \frac{x^i}{i!}$ . What is  $D(0)$ ? Solve your differential equation to find an equation for  $D(x)$ .

(b) Use the equation you found for  $D(x)$  to find an equation for  $d_n$ . Compare this result with the one you computed by inclusion and exclusion.

## C.4 The Product Principle for EGFs

One of our major tools for ordinary generating functions was the product principle. It is thus natural to ask if there is a product principle for exponential generating functions. In Problem 381 you likely found that the EGF for the number of ways of arranging  $n$  books on one shelf was exactly the same as the EGF for the number of permutations of  $[n]$ , namely  $\frac{1}{1-x}$  or  $(1-x)^{-1}$ . Then using our formula from Problem 120 and the ordinary generating function for multisets, you probably found that the EGF for number of ways of arranging  $n$  books on some fixed number  $m$  of bookshelves was  $(1-x)^{-m}$ . Thus the EGF for  $m$  shelves is a product of  $m$  copies of the EGF for one shelf.

○ 387. In Problem 371 what would the exponential generating function have been if we had asked for the number of ways to paint the poles with just one color of paint? With two colors of paint? What is the relationship between the EGF for painting the  $n$  poles with one color of paint and the EGF for painting the  $n$  poles with five colors of paint? What is the relationship among the EGF for painting the  $n$  poles with two colors of paint, the EGF for painting the poles with three colors of paint, and the EGF for painting the poles with five colors of paint?

In Problem 383 you likely found that the EGF for the number of network configurations with  $n$  customers was  $e^{x+x^2/2} = e^x \cdot e^{x^2/2}$ . In Problem 378 you saw that the EGF for the number of permutations on  $n$  elements whose cycle decompositions consist of only one-cycles was  $e^x$ , and in Problem 376 you likely found that the EGF for the number of tennis pairings of  $2n$  people, or equivalently, the number of permutations of  $2n$  objects whose cycle decomposition consists of  $n$  two-cycles is  $e^{x^2/2}$ .

- 388. What can you say about the relationship among the EGF for the number of permutations whose cycle structure consists of disjoint two-cycles and one-cycles, i.e., which are involutions, the exponential generating function for the number of permutations whose cycle decomposition consists of disjoint two-cycles only and the EGF for the number of permutations whose cycle decomposition consists of disjoint one-cycles only (these are identity permutations on their domain)?

In Problem 386 you likely found that the EGF for the number of permutations of  $[n]$  that are derangements is  $\frac{e^{-x}}{1-x}$ . But every permutation is a product of a derangement and a permutation whose cycle decomposition consists of one-cycles, because the permutation that sends  $i$  to  $i$  is a one-cycle, so that when you find the cycle decomposition of a permutation, the cycles of size greater than one are the cycle decomposition of a derangement (of the set of elements moved by the permutation), and the elements not moved by the permutation are one-cycles.

- 389. If we multiply the EGF for derangements times the EGF for the number of permutations whose cycle decompositions consist of one-cycles only, what EGF do we get? For what set of objects have we found the EGF?

We now have four examples in which the EGF for a sequence or a pair of objects is the product of the EGFs for the individual objects making up the sequence or pair.

- 390. What is the coefficient of  $\frac{x^n}{n!}$  in the product of two EGFs  $\sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$  and  $\sum_{j=0}^{\infty} b_j \frac{x^j}{j!}$ ? (A summation sign is appropriate in your answer.)

In the case of painting streetlight poles in Problem 387, let us examine the relationship among the EGF for painting poles with two colors, the EGF for painting poles with three colors, and the EGF for painting poles with five colors,  $e^{5x}$ . To be specific, the EGF for painting poles red and white is

$e^{2x}$  and the EGF for painting poles blue, green, and yellow is  $e^{3x}$ . To decide how to paint poles with red, white, blue, green, and yellow, we can decide which set of poles is to be painted with red and white, and which set of poles is to be painted with blue, green and yellow. Notice that the number of ways to paint a set of poles with red and white depends only on the size of that set, and the number of ways to paint a set of poles with blue, green, and yellow depends only on the size of that set.

- 391. Suppose that  $a_i$  is the number of ways to paint a set of  $i$  poles with red and white, and  $b_j$  is the number of ways to paint a set of  $j$  poles with blue, green and yellow. In how many ways may we take a set  $N$  of  $n$  poles, divide it up into two sets  $I$  and  $J$  (using  $i$  to stand for the size of  $I$  and  $j$  to stand for the size of the set  $J$ , and allowing  $i$  and  $j$  to vary) and paint the poles in  $I$  red and white and the poles in  $J$  blue, green, and yellow? (Don't figure out formulas for  $a_i$  and  $b_j$  to use in your answer; that will make it harder to get the point of the problem!) How does this relate to Problem 390?

Problem 391 shows that the formula you got for the coefficient of  $\frac{x^n}{n!}$  in the product of two EGFs is the formula we get by splitting a set  $N$  of poles into two parts and painting the poles in the first part with red and white and the poles in the second part with blue, green, and yellow. More generally, you could interpret your result in Problem 390 to say that the coefficient of  $\frac{x^n}{n!}$  in the product  $\sum_{i=0}^{\infty} a_i \frac{x^i}{i!} \sum_{j=0}^{\infty} b_j \frac{x^j}{j!}$  of two EGFs is the sum, over all ways of splitting a set  $N$  of size  $n$  into an ordered pair of disjoint sets  $I$  of size  $i$  and  $J$  of size  $j$ , of the product  $a_i b_j$ .

There seem to be two essential features that relate to the product of exponential generating functions. First, we are considering **structures** that consist of a set and some additional mathematical construction on or relationship among the elements of that set. For example, our set might be a set of light poles and the additional construction might be a coloring function defined on that set. Other examples of additional mathematical constructions or relationships on a set could include a permutation of that set; in particular an involution or a derangement, a partition of that set, a graph on that set, a connected graph on that set, an arrangement of the elements of that set around a circle, or an arrangement of the elements of that set on the shelves of a bookcase. In fact a set with no additional construction or arrangement on it is also an example of a structure. When a structure consists of the set  $S$  plus the additional construction, we say the structure *uses*  $S$ . What all the examples we have mentioned in our earlier discussion

of exponential generating functions have in common is that the number of structures that use a given set is determined by the size of that set. We will call a family  $\mathcal{F}$  of structures an *exponential family* of subsets of a set  $X$  or a *species* of structures on subsets of a set  $X$  if structures are defined on finite subsets of  $X$  and if the number of structures in the family using a finite set  $S$  is finite and is determined by the size of  $S$  (that is, if there is a bijection between subsets  $S$  and  $T$  of  $X$ , the number of structures in the family that use  $S$  equals the number of structures in the family that use  $T$ ). We say a structure is an  $\mathcal{F}$ -*structure* if it is a member of the family  $\mathcal{F}$ .

- 392. In Problem 381, why is the family of arrangements of sets of books on a single shelf (assuming they all fit) an exponential family?
- 393. In Problem 383, why is the family of sets of people actually making phone calls (assuming nobody is calling outside the telephone network) at any given time, with the added relationship of who is calling whom, an exponential family? Why is the the family of sets of people who are not using their phones an exponential family (with no additional construction needed)?

The second essential feature of our examples of products of EGFs is that products of EGFs seem to count structures on ordered pairs of two disjoint sets (or more generally on  $k$ -tuples of mutually disjoint sets). For example, we can determine a five coloring of a set  $S$  by partitioning it in all possible ways into two sets and coloring the first set in the pair with our first two colors and our second pair with the last three colors. Or we can partition our set in all possible ways into five parts and color part  $i$  with our  $i$ th color. We don't have to do the same thing to each part of our partition; for example, we could define a derangement on one part and an identity permutation on the other; this defines a permutation on the set we are partitioning, and we have already noted that every permutation arises in this way.

Our combinatorial interpretation of EGFs will involve assuming that the coefficient of  $\frac{x^i}{i!}$  counts the number of structures on a particular set of size  $i$  in an exponential family (or species) of structures on subsets of a set  $X$ . Thus in order to give an interpretation of the product of two EGFs we need to be able to think of ordered pairs of structures on disjoint sets or  $k$ -tuples of structures on disjoint sets as structures themselves. Thus given a structure on a set  $S$  and another structure on a disjoint set  $T$ , we define the ordered pair of structures (which is a mathematical construction!) to be a structure on the set  $S \cup T$ . We call this a *pair structure* on  $S \cup T$ . In

particular, the set of pair structures whose first structure comes from  $\mathcal{F}$  and whose second element comes from  $\mathcal{G}$  is denoted by  $\mathcal{F} \cdot \mathcal{G}$ .

394. Show that if  $\mathcal{F}$  and  $\mathcal{G}$  are exponential families of structures on subsets of a set  $X$ , then the pair structures of  $\mathcal{F} \cdot \mathcal{G}$  form an exponential family (species) of structures.

Given an exponential family (or species)  $\mathcal{F}$  of structures, the number of structures using any particular set of size  $i$  is the same as the number of structures in the family using any other set of size  $i$ . We can thus define the exponential generating function (EGF) for the family as the power series  $\sum_{i=1}^{\infty} a_i \frac{x^i}{i!}$ , where  $a_i$  is the number of structures of  $\mathcal{F}$  that use one particular set of size  $i$ . In Problems 370, 371, 374, 375, 376, 378, 379, 380, 381, 385, and 386 we were computing EGFs for exponential families of subsets of some set.

395. If  $\mathcal{F}$  and  $\mathcal{G}$  are exponential families of subsets of  $X$ , how is the EGF for  $\mathcal{F} \cdot \mathcal{G}$  related to the EGFs for  $\mathcal{F}$  and  $\mathcal{G}$ ? Prove you are right.
396. Without giving the proof, how can you compute the EGF  $f(x)$  for the number of structures using a set of size  $n$  in the exponential family (species)  $\mathcal{F}_1 \cdot \mathcal{F}_2 \cdots \mathcal{F}_k$  of structures on  $k$ -tuples of subsets of  $X$  from the EGFs  $f_i(x)$  for  $\mathcal{F}_i$  for each  $i$  from 1 to  $k$ ? (Here we are using the natural extension of the idea of the pair structure to the idea of a  $k$ -tuple structure.)

The result of Problem 396 will be of enough use to us that we will state it formally along with two useful corollaries.

**Theorem 13** *If  $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$  are exponential families of structures on subsets of a set  $X$  and  $\mathcal{F}_i$  has EGF  $f_i(x)$ , then the family of  $k$ -tuple structures  $\mathcal{F}_1 \cdot \mathcal{F}_2 \cdots \mathcal{F}_k$  has EGF  $\prod_{i=1}^k f_i(x)$ .*

We call Theorem 13 the **General Product Principle for Exponential Generating Functions**. We give two corollaries; the proof of the second is not immediate though not particularly difficult.

**Corollary 3** *If  $\mathcal{F}$  is an exponential family of structures on subsets of  $X$  and  $f(x)$  is the EGF for  $\mathcal{F}$ , then  $f(x)^k$  is the EGF for the  $k$ -tuple structures on  $k$ -tuples of  $\mathcal{F}$ -structures using disjoint subsets of  $X$ .*



Our next corollary uses the idea of a  $k$ -set structure. Suppose we have an exponential family (species)  $\mathcal{F}$  of structures on nonempty subsets of  $X$ , that is, an exponential family of structures which assigns no structures to the empty set. Then we can define a new exponential family (species)  $\mathcal{F}^{(k)}$  of structures, called “ $k$ -set structures,” using nonempty subsets of  $X$ . Given a fixed positive integer  $k$ , a  $k$ -set structure on a subset  $Y$  of  $X$  consists of a  $k$ -element set of nonempty disjoint subsets of  $X$  whose union is  $Y$  and an assignment of an  $\mathcal{F}$ -structure to each of the disjoint subsets. This is an exponential family on the set of subsets of  $X$ ; the subset used by a  $k$ -set structure is the union of the sets of the structure. To recapitulate, the set of  $k$ -set structures on a subset  $Y$  of  $X$  is the set of all possible assignments of  $\mathcal{F}$ -structures to  $k$  nonempty disjoint sets whose union is  $Y$ . (You can also think of the  $k$ -set structures as a family of structures defined on blocks of partitions of subsets of  $X$  into  $k$  blocks.)

**Corollary 4** *If  $\mathcal{F}$  is an exponential family of structures on nonempty subsets of  $X$  and  $f(x)$  is the EGF for  $\mathcal{F}$ , then for each positive integer  $k$ ,  $\frac{f(x)^k}{k!}$  is the EGF for the family  $\mathcal{F}^{(k)}$  of  $k$ -set structures on subsets of  $X$ .*

397. Prove Corollary 4.
- 398. Use the product principle for EGFs to explain the results of Problems 388 and 389.
  - 399. Use the general product principle for EGFs or one of its corollaries to explain the relationship between the EGF for painting streetlight poles in only one color and the EGF for painting streetlight poles in 5 colors in Problems 371 and 387. What is the EGF for the number  $p_n$  of ways to paint  $n$  streetlight poles with some fixed number  $k$  of colors of paint?
  - 400. Use the general product principle for EGFs or its corollary to explain the relationship between the EGF for arranging books on one shelf and the EGF for arranging books on  $n$  shelves in Problem 381.
  - 401. (Optional) Our very first example of exponential generating functions used the binomial theorem to show that the EGF for  $k$ -element permutations of an  $n$  element set is  $(1+x)^n$ . Use the EGF for  $k$ -element permutations of a one-element set and the product principle to prove the same thing. Hint: Review the alternate definition of a function in Section 3.1.2.

402. What is the EGF for the number of ways to paint  $n$  streetlight poles red, white, blue, green and yellow, assuming an even number of poles must be painted green and an even number of poles must be painted yellow? Give a formula for the number of ways to paint  $n$  poles. (Don't forget the factorial!)
- · 403. What is the EGF for the number of functions from an  $n$ -element set onto a one-element set? (Can there be any functions from the empty set onto a one-element set?) What is the EGF for the number  $c_n$  of functions from an  $n$ -element set onto a  $k$  element set (where  $k$  is fixed)? Use this EGF to find an explicit expression for the number of functions from a  $k$ -element set onto an  $n$ -element set and compare the result with what you got by inclusion and exclusion.
- · 404. In Problem 140 you showed that the Bell Numbers  $B_n$  satisfy the equation  $B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_{n-k}$  (or a similar equation for  $B_n$ ). Multiply both sides of this equation by  $\frac{x^n}{n!}$  and sum from  $n = 0$  to infinity. On the left hand side you have a derivative of a certain EGF we might call  $B(x)$ . On the right hand side, you have a product of two EGFs, one of which is  $B(x)$ . What is the other one? What differential equation involving  $B(x)$  does this give you? Solve the differential equation for  $B(x)$ . This is the EGF for the Bell numbers!
- 405. Prove that  $n2^{n-1} = \sum_{k=1}^n \binom{n}{k} k$  by using EGFs.
- 406. In light of Problem 380, why is the EGF for the Stirling numbers  $S(n, k)$  of the second kind (with  $n$  fixed and  $k$  allowed to vary) not  $(e^x - 1)^n$ ? What is it equal to instead?

## C.5 The Exponential Formula

Exponential generating functions turn out to be quite useful in advanced work in combinatorics. One reason why is that it is often possible to give a combinatorial interpretation to the composition of two exponential generating functions. In particular, if  $f(x) = \sum_{i=0}^n a_i \frac{x^i}{i!}$  and  $g(x) = \sum_{j=1}^{\infty} b_j \frac{x^j}{j!}$ , it makes sense to form the composition  $f(g(x))$  because in so doing we need add together only finitely many terms in order to find the coefficient of  $\frac{x^n}{n!}$  in  $f(g(x))$  (since in the EGF  $g(x)$  the dummy variable  $j$  starts at 1). Since our study of combinatorial structures has not been advanced enough to give us applications of a general formula for the compositions of EGFs, we will not

give here the combinatorial interpretation of composition in general. However, we have seen some examples where one particular composition can be applied. Namely, if  $f(x) = e^x = \exp(x)$ , then  $f(g(x)) = \exp(g(x))$  is well defined when  $b_0 = 0$ . We have seen three examples in which an EGF is  $e^{f(x)}$  where  $f(x)$  is another EGF. There is a fourth example in which the exponential function is slightly hidden.

- 407. If  $f(x)$  is the EGF for the number of partitions of an  $n$ -set into one block, and  $g(x)$  is the EGF for the total number of partitions of an  $n$ -element set, that is, for the Bell numbers  $B_n$ , how are the two EGFs related?
- 408. Let  $f(x)$  be the EGF for the number of permutations of an  $n$ -element set with one cycle of size one or two and no other cycles, including no other one-cycles. What is  $f(x)$ ? What is the EGF  $g(x)$  for the number of permutations of an  $n$ -element set all of whose cycles have size one or two, that is, the number of involutions in  $S_n$ , or the number of configurations of a telephone network? How are these two exponential generating functions related?
- 409. Let  $f(x)$  be the EGF for the number of permutations of an  $n$ -element set whose cycle decomposition consists of exactly one two-cycle and no other cycles (this includes having no one-cycles). Let  $g(x)$  be the EGF for the number of permutations whose cycle decomposition consists of two-cycles only, that is, for tennis pairings. What is  $f(x)$ ? What is  $g(x)$ ? How are these two exponential generating functions related?
- 410. Let  $f(x)$  be the EGF for the number of permutations of an  $n$ -element set that have exactly one cycle. Notice that if  $n > 1$  this means they have no one-cycles. (This is the same as the EGF for the number of ways to arrange  $n$  people around a round table.) Let  $g(x)$  be the EGF for the total number of permutations of an  $n$ -element set. What is  $f(x)$ ? What is  $g(x)$ ? How are  $f(x)$  and  $g(x)$  related?

There was one element that our last four problems had in common. In each case our EGF  $f(x)$  involved the number of structures of a certain type (partitions, telephone networks, tennis pairings, permutations) that used only one set of an appropriate kind. (That is, we had a partition with one part, a telephone network consisting either of one person or two people connected to each other, a tennis pairing of one set of two people, or a permutation with one cycle.) Our EGF  $g(x)$  was the number of structures of the same “type” (we put type in quotation marks here because we

don't plan to define it formally) that could consist of any number of sets of the appropriate kind. Notice that the order of these sets was irrelevant. For example, we don't order the blocks of a partition or the cycles in a cycle decomposition of a permutation. Thus we were relating the EGF for structures which were somehow "building blocks" to the EGF for structures which were sets of building blocks. For a reason that you will see later, it is common to call the building blocks *connected* structures. Notice that our connected structures were all based on nonempty sets, so we had no connected structures whose value was the empty set. Thus in each case, if  $f(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$ , we would have  $a_0 = 0$ . The relationship between the EGFs was always  $g(x) = e^{f(x)}$ . We now give a combinatorial explanation for this relationship.

411. Suppose that  $\mathcal{F}$  is an exponential family of structures on subsets of a set  $X$  with no structures on the empty set. Let  $f(x)$  be the EGF for  $\mathcal{F}$ .

(a) In the power series

$$e^{f(x)} = 1 + f(x) + \frac{f(x)^2}{2!} + \cdots + \frac{f(x)^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{f(x)^k}{k!},$$

what does Corollary 4 tell us about the coefficient of  $\frac{x^n}{n!}$  in  $\frac{f(x)^k}{k!}$ ?

(b) What does the coefficient of  $\frac{x^n}{n!}$  in  $e^{f(x)}$  count?

In Problem 411 we proved the following theorem, which is called the **exponential formula**.

**Theorem 14** *Suppose that  $\mathcal{F}$  is an exponential family (species) of structures on subsets of a set  $X$  with no structures on the empty set. Let  $f(x)$  be the EGF for  $\mathcal{F}$ . Then the coefficient of  $\frac{x^n}{n!}$  in  $e^{f(x)}$  is the number of sets of structures on disjoint sets whose union is a particular set of size  $n$ .*

Let us see how the exponential formula applies to the examples in Problems 407, 408, 409 and 410. In Problem 380 our family  $\mathcal{F}$  should consist of one-block partitions of finite subsets of a set, say the set of natural numbers. Since a partition of a set is a set of blocks whose union is  $S$ , a one block partition whose block is  $B$  is the set  $\{B\}$ . Then any nonempty finite subset of the natural numbers is the set used by exactly one structure in  $\mathcal{F}$ . (There is no one block partition of the empty set, so we have no structures using

the empty set.) As you showed in Problem 380 the EGF for partitions with just one block is  $e^x - 1$ . Thus by the exponential formula,  $\exp(e^x - 1)$  is the EGF for sets of disjoint subsets of the positive integers whose union is any particular set  $N$  of size  $n$ . This set of disjoint sets partitions the set  $N$ . Thus  $\exp(e^x - 1)$  is the EGF for partitions of sets of size  $n$ . (As we wrote our description, it is the EGF for partitions of  $n$ -element subsets of the positive integers, but any two  $n$ -element sets have the same number of partitions.) In other words,  $\exp(e^x - 1)$  is the exponential generating function for the Bell numbers  $B_n$ .

- 412. Explain how the exponential formula proves the relationship we saw in Problem 410.
- 413. Explain how the exponential formula proves the relationship we saw in Problem 409.
- 414. Explain how the exponential formula proves the relationship we saw in Problem 408.
- 415. In Problem 371 we saw that the EGF for the number of ways to use five colors of paint to paint  $n$  light poles along the north side of Main Street in Anytown was  $e^{5x}$ . We should expect an explanation of this EGF using the exponential formula. Let  $\mathcal{F}$  be the family of all one-element sets of light poles with the additional construction of an ordered pair consisting of a light pole and a color. Thus a given light pole occurs in five ordered pairs. Put no structure on any other finite set. Show that this is an exponential family of structures on the finite subsets of the positive integers. What is the exponential generating function  $f(x)$  for  $\mathcal{F}$ ? Assuming that there is no upper limit on the number of light poles, what subsets of  $S$  does the exponential formula tell us are counted by the coefficient of  $x^n$  in  $e^{f(x)}$ ? How do the sets being counted relate to ways to paint light poles?

One of the most spectacular applications of the exponential formula is also the reason why, when we regard a combinatorial structure as a set of building block structures, we call the building block structures *connected*. In Chapter 2 we introduced the idea of a connected graph and in Problem 103 we saw examples of graphs which were connected and were not connected. A subset  $C$  of the vertex set of a graph is called a **connected component** of the graph if

- every vertex in  $C$  is connected to every other vertex in that set by a walk whose vertices lie in  $C$ , and
- no other vertex in the graph is connected by a walk to any vertex in  $C$ .

In Problem 239 we showed that each connected component of a graph consists of a vertex and all vertices connected to it by walks in the graph.

- 416. Show that every vertex of a graph lies in one and only one connected component of a graph. (Notice that this shows that the connected components of a graph form a partition of the vertex set of the graph.)
- 417. Explain why no edge of the graph connects two vertices in different connected components.
- 418. Explain why it is that if  $C$  is a connected component of a graph and  $E'$  is the set of all edges of the graph that connect vertices in  $C$ , then the graph with vertex set  $C$  and edge set  $E'$  is a connected graph. We call this graph a *connected component graph* of the original graph.

The last sequence of problems shows that we may think of any graph as the set of its connected component graphs. (Once we know them, we know all the vertices and all the edges of the graph.) Notice that a graph is connected if and only if it has exactly one connected component. Since the connected components form a partition of the vertex set of a graph, the exponential formula will relate the EGF for the number of connected graphs on  $n$  vertices with the EGF for the number of graphs (connected or not) on  $n$  vertices. However, because we can draw as many edges as we want between two vertices of a graph, there are infinitely many graphs on  $n$  vertices, and so the problem of counting them is uninteresting. We can make it interesting by considering **simple graphs**, namely graphs in which each edge has two distinct endpoints and no two edges connect the same two vertices. It is because *connected* simple graphs form the building blocks for viewing all simple graphs as sets of connected components that we refer to the building blocks for structures counted by the EGFs in the exponential formula as *connected* structures.

- •419. Suppose that  $f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}$  is the exponential generating function for the number of simple connected graphs on  $n$  vertices and  $g(x) = \sum_{i=0}^{\infty} a_i \frac{x^i}{i!}$  is the exponential generating function for the number of simple graphs on  $i$  vertices. From this point onward, any use of the word graph means simple graph.

- (a) Is  $f(x) = e^{g(x)}$ , is  $f(x) = e^{g(x)-1}$ , is  $g(x) = e^{f(x)-1}$  or is  $g(x) = e^{f(x)}$ ?
- (b) One of  $a_i$  and  $c_n$  can be computed by recognizing that a simple graph on a vertex set  $V$  is completely determined by its edge set and its edge set is a subset of the set of two-element subsets of  $V$ . Figure out which it is and compute it.
- (c) Write  $g(x)$  in terms of the natural logarithm of  $f(x)$  or  $f(x)$  in terms of the natural logarithm of  $g(x)$ .
- (d) Write  $\log(1 + y)$  as a power series in  $y$ .
- (e) Why is the coefficient of  $\frac{x^0}{0!}$  in  $g(x)$  equal to one? Write  $f(x)$  as a power series in  $g(x) - 1$ .
- (f) You can now use the previous parts of the problem to find a formula for  $c_n$  that involves summing over all partitions of the integer  $n$ . (It isn't the simplest formula in the world, and it isn't the easiest formula in the world to figure out, but it is nonetheless a formula with which one could actually make computations!) Find such a formula.

The point to the last problem is that we can use the exponential formula in reverse to say that if  $g(x)$  is the EGF for the number of (nonempty) connected structures of size  $n$  in a given family of combinatorial structures and  $f(x)$  is the EGF for all the structures of size  $n$  in that family, then not only is  $f(x) = e^{g(x)}$ , but  $g(x) = \ln(f(x))$  as well. Further, if we happen to have a formula for either the coefficients of  $f(x)$  or the coefficients of  $g(x)$ , we can get a formula for the coefficients of the other one!

## C.6 Supplementary Problems

1. Use product principle for EGFs and the idea of coloring a set in two colors to prove the formula  $e^x \cdot e^x = e^{2x}$ .
2. Find the EGF for the number of ordered functions from a  $k$ -element set to an  $n$ -element set.
3. Find the EGF for the number of ways to string  $n$  distinct beads onto a necklace.
4. Find the exponential generating function for the number of broken permutations of a  $k$ -element set into  $n$  parts.

5. Find the EGF for the total number of broken permutations of a  $k$ -element set.
6. Find the EGF for the number of graphs on  $n$  vertices in which every vertex has degree 2.
7. Recall that a cycle of a permutation cannot be empty.
  - (a) What is the EGF for the number of cycles on an even number of elements (i.e. permutations of an even number  $n$  of elements that form an  $n$ -cycle)? Your answer should not have a summation sign in it. Hint: If  $y = \sum_{i=0}^{\infty} \frac{x^{2i}}{2^i}$ , what is the derivative of  $y$ ?
  - (b) What is the EGF for the number of permutations on  $n$  elements whose cycle decomposition consists of even cycles?
  - (c) What is the EGF for the number of cycles on an odd number of elements?
  - (d) What is the EGF for the number of permutations on  $n$  elements whose cycle decomposition consists of odd cycles?
  - (e) How do the EGFs in parts (b) and (d) of this problem relate to the EGF for all permutations on  $n$  elements?



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